






## Solvability of a system of higher order nonlinear difference equations

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### Abstract

In this paper we show that the system of difference equations

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, \quad y_n = \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}},$$

where  $n \in \mathbb{N}_0$ ,  $k$  and  $l$  are positive integers, the parameters  $a, b, c, d, \alpha, \beta, \gamma, \delta$  are real numbers and the initial values  $x_{-j}, y_{-j}, j = \overline{1, k+l}$ , are real numbers, can be solved in the closed form. We also determine the asymptotic behavior of solutions for the case  $l = 1$  and describe the forbidden set of the initial values using the obtained formulas. Our obtained results significantly extend and develop some recent results in the literature.

**Mathematics Subject Classification (2010).** 39A10, 39A20, 39A23, 11B39

**Keywords.** system of difference equations, asymptotic behavior, Fibonacci sequence, forbidden set

### 1. Introduction and preliminaries

For the past two decades there has been an intense interest in nonlinear difference equations, see [2, 5–7, 18, 36, 37, 40, 42, 44]. In the meantime, the two-dimensional or three-dimensional systems of these equations have become the center of interest of researchers. See, for example, [9, 16, 17, 22, 25, 34, 38, 43, 45, 46, 48]. Theoretically, it is very important to characterize the behavior of the solutions of these equations and systems. Although many methods are proposed by researchers, the most basic method to do this is to find a closed formula of the solution of the equation or system and analyze it. In [21], McGrath and Teixeira studied the equation

$$x_{n+1} = \frac{ax_{n-1} + bx_n}{cx_{n-1} + dx_n} x_n, \quad n \in \mathbb{N}_0, \quad (1.1)$$

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Received: 25.10.2018; Accepted: 22.11.2019

where the parameters  $a, b, c, d$ , and the initial values are real numbers. The authors solved Eq. (1.1) and investigated the existence and behavior of the solution of Eq. (1.1) by using some known results. In [39], Tolu et al. considered the following difference equations

$$x_n = \alpha x_{n-k} + \frac{\delta x_{n-k} x_{n-(k+l)}}{\beta x_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0, \quad (1.2)$$

where  $k$  and  $l$  are fixed natural numbers,  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ , and the initial values  $x_{-i}$ ,  $i = \overline{1, k+l}$ , are real numbers. The authors showed that Eq. (1.2) is solvable in closed form and presented formulas for the solutions. They also studied the long-term behavior of the solutions of Eq. (1.2). Some particular cases of the extension of Eq. (1.2) have been studied recently in papers [11–14, 33]. More precisely, Eq. (1.2) in the case  $\alpha = 0, \delta = 1, \beta = 1, \gamma = \pm 1$  was studied in [11, 12] and in [33] the case  $\alpha = 0, \delta = 1$  was studied, while in [13, 14] the case  $k = 1, l = 1$  and  $k = 2, l = 2$  were studied. Eq. (1.1) and Eq. (1.2) actually are particular cases of the higher-order difference equation

$$\frac{x_n}{x_{n-k}} = f\left(\frac{x_{n-l}}{x_{n-k-l}}\right), \quad n \in \mathbb{N}_0, \quad (1.3)$$

where  $k$  and  $l$  are fixed natural numbers. If the equation  $y_n = f(y_{n-l})$ ,  $n \in \mathbb{N}_0$ , is a solvable type, then Eq. (1.3) is solvable, too.

On the other hand, in [41], Eq. (1.2) in the case  $\alpha = 0, \delta = 1, \beta = \pm 1, \gamma = \pm 1$  was extended to the following two-dimensional system of difference equation

$$x_{n+1} = \frac{y_n x_{n-1}}{\pm x_{n-1} \pm y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{\pm y_{n-1} \pm x_n}, \quad n \in \mathbb{N}_0, \quad (1.4)$$

with real nonzero initial values  $x_{-i}, y_{-i}$ , for  $i = 0, 1$ , such that some of their solutions are associated to Fibonacci numbers and some of their solution formulas were proved by induction. However, the formulas have not been confirmed by some theoretical explanations.

In this paper, we give some theoretical explanations for the formulas of solutions of the difference equations system given in (1.4). Moreover, we show that the following more general difference equation system

$$x_n = ay_{n-k} + \frac{dy_{n-k} x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, \quad y_n = \alpha x_{n-k} + \frac{\delta x_{n-k} y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0, \quad (1.5)$$

where  $k$  and  $l$  are positive integers,  $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ , and the initial values  $x_{-j}, y_{-j}$ ,  $j = \overline{1, k+l}$ , are real numbers, can be solved closed form. Also, we investigate some particular cases of system (1.5) and give a study of the long-term behavior of its solutions for the case  $l = 1$ . Finally, we also give natural explanation for the formulas presented in [41].

For more works on the topic, see, for example, [3, 4, 8, 10, 23, 24, 26–32, 35, 47] and the references therein. Also, see the books [1, 15, 20].

Now, we should recall that the Fibonacci sequence  $\{F_n\}_{n=0}^{\infty}$  is defined by

$$F_{n+2} = F_{n+1} + F_n, \quad n \in \mathbb{N}_0, \quad (1.6)$$

with the initial values  $F_0$  and  $F_1$ . Considering [19], the characteristic equation of (1.6) can be clearly obtained as of the form  $x^2 - x - 1 = 0$  having the roots  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$ . Thus, the Binet's Formula for Fibonacci sequence,  $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ , can be thought as a solution of Fibonacci sequence. Also, it is obtained to extend negatively subscripted Fibonacci sequence as

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n, \quad n \in \mathbb{N}_0. \quad (1.7)$$

## 2. Solvability of system (1.5)

In this section we show that the system (1.5) is solvable in closed form. First, we write the system as follows:

$$\frac{x_n}{y_{n-k}} = \frac{ac \frac{y_{n-l}}{x_{n-(k+l)}} + ab + d}{c \frac{y_{n-l}}{x_{n-(k+l)}} + b}, \quad \frac{y_n}{x_{n-k}} = \frac{\alpha\gamma \frac{x_{n-l}}{y_{n-(k+l)}} + \alpha\beta + \delta}{\gamma \frac{x_{n-l}}{y_{n-(k+l)}} + \beta}, \quad n \in \mathbb{N}_0.$$

Putting

$$u_n = \frac{x_n}{y_{n-k}}, \quad v_n = \frac{y_n}{x_{n-k}}, \quad n \geq -l, \quad (2.1)$$

in the last expressions, we get the system of equations

$$u_n = \frac{acv_{n-l} + ab + d}{cv_{n-l} + b}, \quad v_n = \frac{\alpha\gamma u_{n-l} + \alpha\beta + \delta}{\gamma u_{n-l} + \beta}, \quad n \in \mathbb{N}_0, \quad (2.2)$$

where the parameters  $a, b, c, d, \alpha, \beta, \gamma, \delta$ , in the new variables  $u_n$  and  $v_n$ . By using the first equation of (2.2) in its second equation and its second equation in its first equation, we obtain the independent equations

$$u_n = \frac{(ac\alpha\gamma + ab\gamma + d\gamma) u_{n-2l} + ac\alpha\beta + ac\delta + ab\beta + d\beta}{(c\alpha\gamma + b\gamma) u_{n-2l} + c\alpha\beta + c\delta + b\beta}, \quad n \geq l, \quad (2.3)$$

and

$$v_n = \frac{(ac\alpha\gamma + c\alpha\beta + c\delta) v_{n-2l} + ab\alpha\gamma + \alpha\gamma d + \alpha\beta b + \delta b}{(\gamma ac + \beta c) v_{n-2l} + \gamma ab + \gamma d + b\beta}, \quad n \geq l. \quad (2.4)$$

If we apply the decomposition of indexes  $n \rightarrow 2lm+i$ , for  $m \geq -1$  and  $i \in \{l, l+1, \dots, 3l-1\}$ , to (2.3) and (2.4), they become

$$u_{2lm+i} = \frac{(ac\alpha\gamma + ab\gamma + d\gamma) u_{2l(m-1)+i} + ac\alpha\beta + ac\delta + ab\beta + d\beta}{(c\alpha\gamma + b\gamma) u_{2l(m-1)+i} + c\alpha\beta + c\delta + b\beta}, \quad m \in \mathbb{N}_0, \quad (2.5)$$

$$v_{2lm+i} = \frac{(ac\alpha\gamma + c\alpha\beta + c\delta) v_{2l(m-1)+i} + ab\alpha\gamma + \alpha\gamma d + \alpha\beta b + \delta b}{(\gamma ac + \beta c) v_{2l(m-1)+i} + \gamma ab + \gamma d + b\beta}, \quad m \in \mathbb{N}_0. \quad (2.6)$$

Let  $u_m^{(i)} = u_{2lm+i}$ ,  $v_m^{(i)} = v_{2lm+i}$ , for some  $m \geq -1$  and  $i \in \{l, l+1, \dots, 3l-1\}$ . Then equations in (2.5)-(2.6) can be written as the following:

$$u_m^{(i)} = \frac{(ac\alpha\gamma + ab\gamma + d\gamma) u_{m-1}^{(i)} + ac\alpha\beta + ac\delta + ab\beta + d\beta}{(c\alpha\gamma + b\gamma) u_{m-1}^{(i)} + c\alpha\beta + c\delta + b\beta}, \quad m \in \mathbb{N}_0, \quad (2.7)$$

$$v_m^{(i)} = \frac{(ac\alpha\gamma + c\alpha\beta + c\delta) v_{m-1}^{(i)} + ab\alpha\gamma + \alpha\gamma d + \alpha\beta b + \delta b}{(\gamma ac + \beta c) v_{m-1}^{(i)} + \gamma ab + \gamma d + b\beta}, \quad m \in \mathbb{N}_0, \quad (2.8)$$

which is essentially in the form of Riccati difference equation. If we use the change of variables

$$u_m^{(i)} = \frac{ac\alpha\gamma + ab\gamma + d\gamma + c\alpha\beta + c\delta + b\beta}{c\alpha\gamma + b\gamma} r_m - \frac{c\alpha\beta + c\delta + b\beta}{c\alpha\gamma + b\gamma}, \quad m \geq -1, \quad (2.9)$$

where  $c\alpha\gamma + b\gamma \neq 0$ , in Eq. (2.7) and the change of variables

$$v_m^{(i)} = \frac{ac\alpha\gamma + ab\gamma + d\gamma + c\alpha\beta + c\delta + b\beta}{\gamma ac + \beta c} s_m - \frac{\gamma ab + \gamma d + b\beta}{\gamma ac + \beta c}, \quad m \geq -1, \quad (2.10)$$

where  $\gamma ac + \beta c \neq 0$ , in Eq. (2.8), then (2.7) and (2.8) are transformed into the following equations:

$$r_m = 1 - \frac{R}{r_{m-1}}, \quad s_m = 1 - \frac{R}{s_{m-1}}, \quad m \in \mathbb{N}_0, \quad (2.11)$$

where  $aca\gamma + ab\gamma + d\gamma + ca\beta + c\delta + b\beta \neq 0$  and  $R = \frac{cd\gamma\delta}{(aca\gamma + ab\gamma + d\gamma + ca\beta + c\delta + b\beta)^2}$ , respectively. The equations in (2.11) can be transformed into the following equations:

$$z_{m+1} = z_m - Rz_{m-1}, m \in \mathbb{N}_0, \quad (2.12)$$

$$\widehat{z}_{m+1} = \widehat{z}_m - R\widehat{z}_{m-1}, m \in \mathbb{N}_0, \quad (2.13)$$

by means of the changes of variables  $r_m = \frac{z_{m+1}}{z_m}$  with the initial values  $z_{-1} = 1$  and  $z_0 = r_{-1}$  and  $s_m = \frac{\widehat{z}_{m+1}}{\widehat{z}_m}$  with the initial values  $\widehat{z}_{-1} = 1$  and  $\widehat{z}_0 = s_{-1}$ , respectively. (2.12) and (2.13) are in the same form and have the characteristic equation  $\lambda^2 - \lambda + R = 0$ . If  $\lambda_1$  and  $\lambda_2$  are the roots of the characteristic equations, namely  $\lambda_1 = \frac{1+\sqrt{1-4R}}{2}$  and  $\lambda_2 = \frac{1-\sqrt{1-4R}}{2}$ , the general solutions of equations in (2.12) and (2.13) are

$$z_m = \begin{cases} \frac{(\lambda_1 r_{-1} - R)\lambda_1^m - (\lambda_2 r_{-1} - R)\lambda_2^m}{\lambda_1 - \lambda_2} & \text{if } R \neq \frac{1}{4}, \\ \left(\frac{2r_{-1} + (2r_{-1} - 1)m}{2}\right) \left(\frac{1}{2}\right)^m & \text{if } R = \frac{1}{4}, \end{cases} m \geq -1, \quad (2.14)$$

$$\widehat{z}_m = \begin{cases} \frac{(\lambda_1 s_{-1} - R)\lambda_1^m - (\lambda_2 s_{-1} - R)\lambda_2^m}{\lambda_1 - \lambda_2} & \text{if } R \neq \frac{1}{4}, \\ \left(\frac{2s_{-1} + (2s_{-1} - 1)m}{2}\right) \left(\frac{1}{2}\right)^m & \text{if } R = \frac{1}{4}, \end{cases} m \geq -1, \quad (2.15)$$

where  $R = \frac{cd\gamma\delta}{(aca\gamma + ab\gamma + d\gamma + ca\beta + c\delta + b\beta)^2}$ . By substituting (2.14) into  $r_m = \frac{z_{m+1}}{z_m}$  and (2.15) into  $s_m = \frac{\widehat{z}_{m+1}}{\widehat{z}_m}$ , we have

$$r_m = \begin{cases} \frac{(\lambda_1 r_{-1} - R)\lambda_1^{m+1} - (\lambda_2 r_{-1} - R)\lambda_2^{m+1}}{(\lambda_1 r_{-1} - R)\lambda_1^m - (\lambda_2 r_{-1} - R)\lambda_2^m} & \text{if } R \neq \frac{1}{4}, \\ \frac{2r_{-1} + (2r_{-1} - 1)(m+1)}{4r_{-1} + (4r_{-1} - 2)m} & \text{if } R = \frac{1}{4}, \end{cases} m \geq -1, \quad (2.16)$$

$$s_m = \begin{cases} \frac{(\lambda_1 s_{-1} - R)\lambda_1^{m+1} - (\lambda_2 s_{-1} - R)\lambda_2^{m+1}}{(\lambda_1 s_{-1} - R)\lambda_1^m - (\lambda_2 s_{-1} - R)\lambda_2^m} & \text{if } R \neq \frac{1}{4}, \\ \frac{2s_{-1} + (2s_{-1} - 1)(m+1)}{4s_{-1} + (4s_{-1} - 2)m} & \text{if } R = \frac{1}{4}, \end{cases} m \geq -1, \quad (2.17)$$

respectively. Consequently, we get

$$u_m^{(i)} = \begin{cases} \frac{A}{B_1} \frac{(\lambda_1 B_1 u_{-1}^{(i)} + \lambda_1 C_1 - RA)\lambda_1^{m+1} - (\lambda_2 B_1 u_{-1}^{(i)} + \lambda_2 C_1 - RA)\lambda_2^{m+1}}{(\lambda_1 B_1 u_{-1}^{(i)} + \lambda_1 C_1 - RA)\lambda_1^m - (\lambda_2 B_1 u_{-1}^{(i)} + \lambda_2 C_1 - RA)\lambda_2^m} - \frac{C_1}{B_1} & \text{if } R \neq \frac{1}{4}, \\ \frac{A}{B_1} \left( \frac{2B_1 u_{-1}^{(i)} + 2C_1 + (2B_1 u_{-1}^{(i)} + 2C_1 - A)(m+1)}{4B_1 u_{-1}^{(i)} + 4C_1 + (4B_1 u_{-1}^{(i)} + 4C_1 - 2A)m} \right) - \frac{C_1}{B_1} & \text{if } R = \frac{1}{4}, \end{cases} \quad (2.18)$$

and

$$v_m^{(i)} = \begin{cases} \frac{A}{B_2} \frac{(\lambda_1 B_2 v_{-1}^{(i)} + \lambda_1 C_2 - RA)\lambda_1^{m+1} - (\lambda_2 B_2 v_{-1}^{(i)} + \lambda_2 C_2 - RA)\lambda_2^{m+1}}{(\lambda_1 B_2 v_{-1}^{(i)} + \lambda_1 C_2 - RA)\lambda_1^m - (\lambda_2 B_2 v_{-1}^{(i)} + \lambda_2 C_2 - RA)\lambda_2^m} - \frac{C_2}{B_2} & \text{if } R \neq \frac{1}{4}, \\ \frac{A}{B_2} \left( \frac{2B_2 v_{-1}^{(i)} + 2C_2 + (2B_2 v_{-1}^{(i)} + 2C_2 - A)(m+1)}{4B_2 v_{-1}^{(i)} + 4C_2 + (4B_2 v_{-1}^{(i)} + 4C_2 - 2A)m} \right) - \frac{C_2}{B_2} & \text{if } R = \frac{1}{4}, \end{cases} \quad (2.19)$$

and so

$$u_{2lm+i} = \begin{cases} \frac{A}{B_1} \frac{G_1 \lambda_1^{m+1} - G_2 \lambda_2^{m+1}}{G_1 \lambda_1^m - G_2 \lambda_2^m} - \frac{C_1}{B_1} & \text{if } R \neq \frac{1}{4}, \\ \frac{A}{B_1} \left( \frac{H_1 + (H_1 - A)(m+1)}{2H_1 + (2H_1 - 2A)m} \right) - \frac{C_1}{B_1} & \text{if } R = \frac{1}{4}, \end{cases} \quad (2.20)$$

and

$$v_{2lm+i} = \begin{cases} \frac{A}{B_2} \frac{K_1 \lambda_1^{m+1} - K_2 \lambda_2^{m+1}}{K_1 \lambda_1^m - K_2 \lambda_2^m} - \frac{C_2}{B_2} & \text{if } R \neq \frac{1}{4}, \\ \frac{A}{B_2} \left( \frac{H_2 + (H_2 - A)(m+1)}{2H_2 + (2H_2 - 2A)m} \right) - \frac{C_2}{B_2} & \text{if } R = \frac{1}{4}, \end{cases} \quad (2.21)$$

where  $A = aca\gamma + ab\gamma + d\gamma + ca\beta + c\delta + b\beta$ ,  $B_1 = ca\gamma + b\gamma$ ,  $C_1 = ca\beta + c\delta + b\beta$ ,  $B_2 = \gamma ac + \beta c$ ,  $C_2 = \gamma ab + \gamma d + \beta b$ ,  $G_1 = \lambda_1 B_1 \frac{x_{i-2l}}{y_{i-2l-k}} + \lambda_1 C_1 - RA$ ,  $G_2 = \lambda_2 B_1 \frac{x_{i-2l}}{y_{i-2l-k}} + \lambda_2 C_1 - RA$ ,

$K_1 = \lambda_1 B_2 \frac{y_{i-2l}}{x_{i-2l-k}} + \lambda_1 C_2 - RA$ ,  $K_2 = \lambda_2 B_2 \frac{y_{i-2l}}{x_{i-2l-k}} + \lambda_2 C_2 - RA$ ,  $H_1 = 2B_1 \frac{x_{i-2l}}{y_{i-2l-k}} + 2C_1$ ,  $H_2 = 2B_2 \frac{y_{i-2l}}{x_{i-2l-k}} + 2C_2$  for  $i \in \{l, l+1, \dots, 3l-1\}$ . From (2.1) we have that

$$x_n = u_n y_{n-k} = u_n v_{n-k} x_{n-2k}, \quad y_n = v_n x_{n-k} = v_n u_{n-k} y_{n-2k}, \quad n \geq k-l, \tag{2.22}$$

$$x_{2km+i} = u_{2km+i} y_{2km+i-k} = u_{2km+i} v_{2km+i-k} x_{2k(m-1)+i}, \quad m \in \mathbb{N}_0, \tag{2.23}$$

$$y_{2km+i} = v_{2km+i} x_{2km+i-k} = v_{2km+i} u_{2km+i-k} y_{2k(m-1)+i}, \quad m \in \mathbb{N}_0, \tag{2.24}$$

from which it follows that

$$x_{2km+i} = x_{i-2k} \prod_{j=0}^m u_{2kj+i} v_{2kj+i-k}, \tag{2.25}$$

and

$$y_{2km+i} = y_{i-2k} \prod_{j=0}^m v_{2kj+i} u_{2kj+i-k}, \tag{2.26}$$

for every  $m \in \mathbb{N}_0$  and  $i = \overline{k-l, 3k-l-1}$ . Since every non-negative integer can be written in the form  $lm_1 + j$ , where  $m_1 \in \mathbb{N}_0$  and  $j \in \{0, 1, \dots, l-1\}$ , we get that

$$x_{2klm_1+2kj+i} = x_{2kj+i-2k} \prod_{s=0}^{lm_1} u_{2ks+2kj+i} v_{2ks+2kj+i-k}, \quad m_1 \in \mathbb{N}_0, \tag{2.27}$$

$$y_{2klm_1+2kj+i} = y_{2kj+i-2k} \prod_{s=0}^{lm_1} v_{2ks+2kj+i} u_{2ks+2kj+i-k}, \quad m_1 \in \mathbb{N}_0, \tag{2.28}$$

where  $j \in \{0, 1, \dots, l-1\}$  and  $i = \overline{k-l, 3k-l-1}$ .

By the following theorem, we characterize the forbidden set of the initial values for the system (1.5).

**Theorem 2.1.** *The forbidden set of the initial values for system (1.5) is the union of two sets*

$$\left\{ \vec{X} : x_{-j} = 0 \text{ or } y_{-j} = 0, j = \overline{1, k} \right\}$$

and

$$\bigcup_{m \in \mathbb{N}_0} \bigcup_{j=1}^{k+l} \left\{ \vec{X} : \frac{y_{j-l}}{x_{j-k-l}} = (f \circ g)^{-m} \left( -\frac{b}{c} \right) \text{ or } \frac{x_{j-l}}{y_{j-k-l}} = (g \circ f)^{-m} (K_1) \right. \\ \left. \text{or } (g \circ f)^{-m} \left( -\frac{\beta}{\gamma} \right) \text{ or } (f \circ g)^{-m} (K_2) \right\},$$

where  $\vec{X} = (x_{-k-l}, x_{-k-l+1}, \dots, x_{-1}, y_{-k-l}, y_{-k-l+1}, \dots, y_{-1})$ ,  $K_1 = -\frac{b\beta+c\alpha\beta+c\delta}{b\gamma+c\alpha\gamma}$ ,  $K_2 = -\frac{b\beta+ab\gamma+d\gamma}{c\beta+ac\gamma}$ .

**Proof.** Let  $(x_n, y_n)_{n \geq -k-l}$  be a solution of the system (1.5). If  $x_{-j} = 0$  or  $y_{-j} = 0$  for some  $j = \overline{1, k}$ , then  $x_n$  cannot be calculated after a term  $y_{n_0}$ ,  $n_0 \in \mathbb{N}_0$ . For example, if  $x_{-k} = 0$ , then  $y_0 = 0$ , and so  $x_l$  cannot be calculated. For the dual of this case, the result is same, too. That is, if  $y_{-k} = 0$ , then  $x_0 = 0$ , and so  $y_l$  cannot be calculated. For the other initial values, the case is not same. Because, if  $x_{-j} = 0$  or  $y_{-j} = 0$  for some  $j = \overline{k+1, k+l}$ , then  $x_n \neq 0$ ,  $y_n \neq 0$  for  $n \geq 0$ . So, we incorporate the set

$$\left\{ \vec{X} : x_{-j} = 0 \text{ or } y_{-j} = 0, j = \overline{1, k} \right\}$$

in the forbidden set. Now, we suppose that  $x_n \neq 0$  and  $y_n \neq 0$ . The solution  $(x_n, y_n)_{n \geq -k-l}$  of the system (1.5) is not defined if and only if  $bx_{n-(k+l)} + cy_{n-l} = 0$  and  $\beta y_{n-(k+l)} + \gamma x_{n-l} =$

0 which correspond to the statements  $\frac{y_{n-l}}{x_{n-k-l}} = -\frac{b}{c}$  and  $\frac{x_{n-l}}{y_{n-k-l}} = -\frac{\beta}{\gamma}$  for  $n \geq -k-l$ , respectively. Therefore, by taking into account (2.1), we have

$$v_{n-l} = -\frac{b}{c} \text{ and } u_{n-l} = -\frac{\beta}{\gamma} \quad (2.29)$$

for  $n \in \mathbb{N}_0$ . Now, we again consider the system (2.2) and the functions

$$g(t) = \frac{act + ab + d}{ct + b}, \quad f(t) = \frac{\alpha\gamma t + \alpha\beta + \delta}{\gamma t + \beta}$$

which correspond to the equations of (2.2). Hence, we can describe the solutions of (2.7) and (2.8) as follows:

$$v_{2m-1}^{(i)} = (f \circ g)^m \left( v_{-1}^{(i)} \right), \quad (2.30)$$

$$v_{2m}^{(i)} = ((f \circ g)^m \circ f) \left( u_{-1}^{(i)} \right), \quad (2.31)$$

$$u_{2m-1}^{(i)} = (g \circ f)^m \left( u_{-1}^{(i)} \right), \quad (2.32)$$

$$u_{2m}^{(i)} = ((g \circ f)^m \circ g) \left( v_{-1}^{(i)} \right), \quad (2.33)$$

for  $m \geq 0$ . By using (2.29) and the implicit forms (2.30)-(2.33), we have

$$v_{-1}^{(i)} = (f \circ g)^{-m} \left( -\frac{b}{c} \right), \quad (2.34)$$

$$u_{-1}^{(i)} = (g \circ f)^{-m} \left( f^{-1} \left( -\frac{b}{c} \right) \right) = (g \circ f)^{-m} \left( -\frac{b\beta + c\alpha\beta + c\delta}{b\gamma + c\alpha\gamma} \right), \quad (2.35)$$

$$u_{-1}^{(i)} = (g \circ f)^{-m} \left( -\frac{\beta}{\gamma} \right), \quad (2.36)$$

$$v_{-1}^{(i)} = (f \circ g)^{-m} \left( g^{-1} \left( -\frac{\beta}{\gamma} \right) \right) = (f \circ g)^{-m} \left( -\frac{b\beta + ab\gamma + d\gamma}{c\beta + ac\gamma} \right), \quad (2.37)$$

where

$$g^{-1}(t) = \frac{-bt + ab + d}{ct - ac}, \quad f^{-1}(t) = \frac{-\beta t + \alpha\beta + \delta}{\gamma t - \alpha\gamma},$$

respectively. This means that if one of the conditions in (2.34)-(2.37) holds, then  $2m - th$  iteration or  $(2m + 1) - th$  iteration in (2.2) cannot be calculated. Consequently, desired result follows from (2.1). Also, note that system associated with the functions  $f^{-1}$  and  $g^{-1}$  is

$$w_n = \frac{-b\tilde{w}_{n-l} + ab + d}{c\tilde{w}_{n-l} - ac}, \quad \tilde{w}_n = \frac{-\beta w_{n-l} + \alpha\beta + \delta}{\gamma w_{n-l} - \alpha\gamma}, \quad n \in \mathbb{N}_0,$$

and is solvable. That is, the right hand sides of the equalities in (2.34)-(2.37) can be obtained in the closed form.  $\square$

### 3. Some special cases of the system (1.5)

In this section we deal with some special cases of the system (1.5). We note that the system (1.5) is trivial, if  $a = d = 0$  or  $\alpha = \delta = 0$  and can be undefined, if  $b = c = 0$  or  $\beta = \gamma = 0$ . So, we consider definable cases.

### 3.1. Case $d = 0$

If  $d = 0$  then the system (1.5) reduces to the following system

$$x_n = ay_{n-k}, \quad y_n = \alpha x_{n-k} + \frac{\delta x_{n-k} y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0.$$

By using the changes of variables in (2.1) we get

$$u_n = a, \quad v_n = \frac{\alpha\gamma a + \alpha\beta + \delta}{\gamma a + \beta}. \quad (3.1)$$

Putting (3.1) in (2.27) and (2.28), we get solutions of the system (1.5).

### 3.2. Case $\delta = 0$

If  $\delta = 0$  then the system (1.5) reduces to the following system

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, \quad y_n = \alpha x_{n-k}, \quad n \in \mathbb{N}_0.$$

This case is dual of previous case. By using the changes of variables in (2.1) we get

$$u_n = \frac{ac\alpha + ab + d}{c\alpha + b}, \quad v_n = \alpha \quad (3.2)$$

Putting (3.2) in (2.27) and (2.28), we get solutions of the system (1.5).

### 3.3. Case $d = \delta = 0$

If  $d = \delta = 0$  then the system (1.5) reduces to the following linear system

$$x_n = ay_{n-k}, \quad y_n = \alpha x_{n-k}, \quad n \in \mathbb{N}_0,$$

which is one of simplest cases. By using the changes of variables in (2.1) we get

$$u_n = a, \quad v_n = \alpha \quad (3.3)$$

Putting (3.3) in (2.27) and (2.28), we get solutions of the system (1.5).

### 3.4. Case $c = 0$

If  $c = 0$  then the system (1.5) reduces to the following system

$$x_n = \left(\frac{ab+d}{b}\right)y_{n-k}, \quad y_n = \alpha x_{n-k} + \frac{\delta x_{n-k} y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0.$$

By using the changes of variables in (2.1) we get

$$u_n = \frac{ab+d}{b}, \quad v_n = \frac{\alpha\beta b + \alpha\gamma ab + \alpha\gamma d + b\delta}{\beta b + \gamma ab + \gamma d}. \quad (3.4)$$

Putting (3.4) in (2.27) and (2.28), we get solutions of the system (1.5).

### 3.5. Case $\gamma = 0$

If  $\gamma = 0$  then the system (1.5) reduces to the following system

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, \quad y_n = \left(\frac{\alpha\beta + \delta}{\beta}\right)x_{n-k}, \quad n \in \mathbb{N}_0.$$

By using the changes of variables in (2.1) we get

$$u_n = \frac{ab\beta + ac\alpha\beta + ac\delta + \beta d}{b\beta + c\alpha\beta + c\delta}, \quad v_n = \frac{\alpha\beta + \delta}{\beta}. \quad (3.5)$$

Putting (3.5) in (2.27) and (2.28), we get solutions of the system (1.5).

**3.6. Case  $c = \gamma = 0$** 

If  $c = \gamma = 0$  then the system (1.5) reduces to the following system

$$x_n = \left(\frac{ab+d}{b}\right) y_{n-k}, \quad y_n = \left(\frac{\alpha\beta+\delta}{\beta}\right) x_{n-k}, \quad n \in \mathbb{N}_0.$$

By using the changes of variables in (2.1) we get

$$u_n = \frac{ab+d}{b}, \quad v_n = \frac{\alpha\beta+\delta}{\beta}. \quad (3.6)$$

Putting (3.6) in (2.27) and (2.28), we get solutions of the system (1.5).

**3.7. Case  $a = c = 0$** 

If  $a = c = 0$  then the system (1.5) reduces to the following system

$$x_n = \left(\frac{d}{b}\right) y_{n-k}, \quad y_n = \alpha x_{n-k} + \frac{\delta x_{n-k} y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0.$$

By using the changes of variables in (2.1) we get

$$u_n = \frac{d}{b}, \quad v_n = \frac{\alpha\gamma d + \alpha\beta b + \delta b}{\gamma d + \beta b}. \quad (3.7)$$

Putting (3.7) in (2.27) and (2.28), we get solutions of the system (1.5).

**3.8. Case  $\alpha = \gamma = 0$** 

If  $\alpha = \gamma = 0$  then the system (1.5) reduces to the following system

$$x_n = a y_{n-k} + \frac{d y_{n-k} x_{n-(k+l)}}{b x_{n-(k+l)} + c y_{n-l}}, \quad y_n = \frac{\delta}{\beta} x_{n-k}, \quad n \in \mathbb{N}_0.$$

By using the changes of variables in (2.1) we get

$$u_n = \frac{ac\delta + ab\beta + d\beta}{c\delta + b\beta}, \quad v_n = \frac{\delta}{\beta}. \quad (3.8)$$

Putting (3.8) in (2.27) and (2.28), we get solutions of the system (1.5).

**3.9. Case  $a = c = \alpha = \gamma = 0$** 

If  $a = c = \alpha = \gamma = 0$  then the system (1.5) reduces to the following system

$$x_n = \frac{d}{b} y_{n-k}, \quad y_n = \frac{\delta}{\beta} x_{n-k}, \quad n \in \mathbb{N}_0.$$

By using the changes of variables in (2.1), we get

$$u_n = \frac{d}{b}, \quad v_n = \frac{\delta}{\beta}. \quad (3.9)$$

Putting (3.9) in (2.27) and (2.28), we get solutions of the system (1.5).



**3.10. Case  $a = b = 0$**

If  $a = b = 0$  then the system (1.5) reduces to the following system

$$x_n = \frac{dy_{n-k}x_{n-(k+l)}}{cy_{n-l}}, y_n = \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, n \in \mathbb{N}_0. \tag{3.10}$$

By using the changes of variables in (2.1), we get  $u_n = \frac{d}{cv_{n-l}}$  from the first equation of (3.10) and so

$$v_n = \frac{(\alpha\beta c + \delta c)v_{n-2l} + \alpha\gamma d}{\beta cv_{n-2l} + \gamma d}, n \geq l, \tag{3.11}$$

from its second equation. Eq. (3.11) can be solved and its solution is given by

$$v_{2lm+i} = \begin{cases} \frac{A_{10}}{B_{10}} \frac{P_{101}\lambda_1^{m+1} - P_{102}\lambda_2^{m+1}}{P_{101}\lambda_1^m - P_{102}\lambda_2^m} - \frac{C_{10}}{B_{10}} & \text{if } R_{10} \neq \frac{1}{4}, \\ \frac{A_{10}}{B_{10}} \left( \frac{P_{10} + (P_{10} - A_{10})(m+1)}{2P_{10} + (2P_{10} - 2A_{10})m} \right) - \frac{C_{10}}{B_{10}} & \text{if } R_{10} = \frac{1}{4}, \end{cases} \tag{3.12}$$

where  $R_{10} = \frac{\delta c \gamma d}{(\alpha\beta c + \delta c + \gamma d)^2}$ ,  $A_{10} = \alpha\beta c + \delta c + \gamma d$ ,  $B_{10} = \beta c$  and  $C_{10} = \gamma d$ ,  $P_{10j} = \lambda_j B_{10} \frac{y_{i-2l}}{x_{i-2l-k}} + \lambda_j C_{10} - R_{10} A_{10}$ ,  $P_{10} = 2B_{10} \frac{y_{i-2l}}{x_{i-2l-k}} + 2C_{10}$  for  $j \in \{1, 2\}$ ,  $i = \overline{l, 3l-1}$ . From the equation  $u_n = \frac{d}{cv_{n-l}}$ , we get

$$u_{2m+i_1} = \frac{d}{cv_{2m+i_1-l}}. \tag{3.13}$$

for  $2m \geq -i_1$  and so

$$u_{2ln+2r+i_1} = \frac{d}{cv_{l(2n-1)+2r+i_1}}. \tag{3.14}$$

for every  $n \in \mathbb{N}_0$ ,  $r = \overline{0, l-1}$  and  $i_1 \in \{0, 1\}$ . From (3.14), we get

$$u_{2ln+2r+i_1} = \begin{cases} \frac{d}{c} \frac{\frac{1}{\frac{A_{10}}{B_{10}} \frac{\widehat{P}_{101}\lambda_1^n - \widehat{P}_{102}\lambda_2^n}{\widehat{P}_{101}\lambda_1^{n-1} - \widehat{P}_{102}\lambda_2^{n-1}} - \frac{C_{10}}{B_{10}}}}{1} & \text{if } R_{10} \neq \frac{1}{4}, \\ \frac{d}{c} \frac{\frac{1}{\frac{A_{10}}{B_{10}} \left( \frac{\widehat{P}_{10} + (\widehat{P}_{10} - A_{10})n}{2\widehat{P}_{10} + (2\widehat{P}_{10} - 2A_{10})(n-1)} \right) - \frac{C_{10}}{B_{10}}}}{1} & \text{if } R_{10} = \frac{1}{4}. \end{cases} \tag{3.15}$$

where  $R_{10} = \frac{\delta c \gamma d}{(\alpha\beta c + \delta c + \gamma d)^2}$ ,  $A_{10} = \alpha\beta c + \delta c + \gamma d$ ,  $B_{10} = \beta c$  and  $C_{10} = \gamma d$ ,  $\widehat{P}_{10j} = \lambda_j B_{10} \frac{y_{i-2l}}{x_{i-2l-k}} + \lambda_j C_{10} - R_{10} A_{10}$ ,  $\widehat{P}_{10} = 2B_{10} \frac{y_{i-2l}}{x_{i-2l-k}} + 2C_{10}$  for  $r = \overline{0, l-1}$ ,  $i_1 \in \{0, 1\}$ ,  $j \in \{1, 2\}$ ,  $i = \overline{l, 3l-1}$ . Putting (3.12) and (3.15) in (2.27) and (2.28), we get solutions of the system (1.5).

**3.11. Case  $\alpha = \beta = 0$**

This case is dual of the previous case and reduces the system (1.5) to the following system

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, y_n = \frac{\delta x_{n-k}y_{n-(k+l)}}{\gamma x_{n-l}}, n \in \mathbb{N}_0. \tag{3.16}$$

By using the changes of variables in (2.1), the system (3.16) is transformed into the system

$$u_n = \frac{(ab\gamma + d\gamma)u_{n-2l} + ac\delta}{b\gamma u_{n-2l} + c\delta}, n \geq l, v_n = \frac{\delta}{\gamma u_{n-l}}, n \in \mathbb{N}_0. \tag{3.17}$$

The first equation of the system (3.17) can be solved and its solution is given by

$$u_{2lm+i} = \begin{cases} \frac{A_{11}}{B_{11}} \frac{P_{111}\lambda_1^{m+1} - P_{112}\lambda_2^{m+1}}{P_{111}\lambda_1^m - P_{112}\lambda_2^m} - \frac{C_{11}}{B_{11}} & \text{if } R_{11} \neq \frac{1}{4}, \\ \frac{A_{11}}{B_{11}} \left( \frac{P_{11} + (P_{11} - A_{11})(m+1)}{2P_{11} + (2P_{11} - 2A_{11})m} \right) - \frac{C_{11}}{B_{11}} & \text{if } R_{11} = \frac{1}{4}, \end{cases} \tag{3.18}$$

where  $R_{11} = \frac{d\gamma c\delta}{(ab\gamma+d\gamma+c\delta)^2}$ ,  $A_{11} = ab\gamma + d\gamma + c\delta$ ,  $B_{11} = b\gamma$  and  $C_{11} = c\delta$ ,  $P_{11j} = \lambda_j B_{11} \frac{x_{i-2l}}{y_{i-2l-k}} + \lambda_j C_{11} - R_{11} A_{11}$ ,  $P_{11} = 2B_{11} \frac{x_{i-2l}}{y_{i-2l-k}} + 2C_{11}$  for  $j \in \{1, 2\}$ ,  $i = \overline{l, 3l-1}$ . From second equation of the system (3.17), we get

$$v_{2m+i_1} = \frac{\delta}{\gamma u_{2m+i_1-l}}, \quad (3.19)$$

for  $2m \geq -i_1$  and so

$$v_{2ln+2r+i_1} = \frac{\delta}{\gamma u_{l(2n-1)+2r+i_1}}, \quad (3.20)$$

for every  $n \in \mathbb{N}_0$ ,  $r = \overline{0, l-1}$  and  $i_1 \in \{0, 1\}$ . From (3.20), we get the formulas

$$v_{2ln+2r+i_1} = \begin{cases} \frac{\delta}{\gamma} \frac{\frac{A_{11}}{B_{11}} \frac{1}{\widehat{P}_{111}\lambda_1^n - \widehat{P}_{112}\lambda_2^n} - \frac{C_{11}}{B_{11}}} & \text{if } R_{11} \neq \frac{1}{4}, \\ \frac{\delta}{\gamma} \frac{\frac{A_{11}}{B_{11}} \left( \frac{1}{2\widehat{P}_{11} + (2\widehat{P}_{11} - 2A_{11})(n-1)} \right) - \frac{C_{11}}{B_{11}}} & \text{if } R_{11} = \frac{1}{4}, \end{cases} \quad (3.21)$$

where  $R_{11} = \frac{d\gamma c\delta}{(ab\gamma+d\gamma+c\delta)^2}$ ,  $A_{11} = ab\gamma + d\gamma + c\delta$ ,  $B_{11} = b\gamma$  and  $C_{11} = c\delta$ ,  $\widehat{P}_{11j} = \lambda_j B_{11} \frac{x_{i-2l}}{y_{i-2l-k}} + \lambda_j C_{11} - R_{11} A_{11}$ ,  $\widehat{P}_{11} = 2B_{11} \frac{x_{i-2l}}{y_{i-2l-k}} + 2C_{11}$  for  $r = \overline{0, l-1}$ ,  $i_1 \in \{0, 1\}$ ,  $j \in \{1, 2\}$ ,  $i = \overline{l, 3l-1}$ . Putting (3.18) and (3.21) in (2.27) and (2.28), we get solutions of the system (1.5).

### 3.12. Case $a = b = \alpha = \beta = 0$

In this case, the system (1.5) reduces to the following system

$$x_n = \frac{dy_{n-k}x_{n-(k+l)}}{cy_{n-l}}, \quad y_n = \frac{\delta x_{n-k}y_{n-(k+l)}}{\gamma x_{n-l}}, \quad n \in \mathbb{N}_0.$$

By using the changes of variables in (2.1), we get the  $2l$ -order linear difference equations

$$u_n = \frac{d\gamma}{c\delta} u_{n-2l}, \quad v_n = \frac{c\delta}{\gamma d} v_{n-2l}, \quad n \geq l, \quad (3.22)$$

after some simple operations. From which it follows that

$$u_{2lm+i} = \left( \frac{d\gamma}{c\delta} \right)^{m+1} \frac{x_{i-2l}}{y_{i-k-2l}}, \quad v_{2lm+i} = \left( \frac{c\delta}{\gamma d} \right)^{m+1} \frac{y_{i-2l}}{x_{i-k-2l}}, \quad (3.23)$$

where  $m \in \mathbb{N}_0$  and  $i \in \{l, l+1, \dots, 3l-1\}$ . Putting (3.23) in (2.27) and (2.28), we get solutions of the system (1.5).

### 3.13. Case $b = 0$

If  $b = 0$ , then the system (1.5) reduces to the following system

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{cy_{n-l}}, \quad y_n = \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0. \quad (3.24)$$

By applying (2.1) to (3.24), we get the system

$$u_n = \frac{acv_{n-l} + d}{cv_{n-l}}, \quad v_n = \frac{\alpha\gamma u_{n-l} + \alpha\beta + \delta}{\gamma u_{n-l} + \beta}, \quad n \in \mathbb{N}_0. \quad (3.25)$$

From (3.25), we get the equations

$$\begin{aligned} u_n &= \frac{(aca\gamma + d\gamma)u_{n-2l} + aca\beta + ac\delta + \beta d}{ca\gamma u_{n-2l} + ca\beta + c\delta}, \\ v_n &= \frac{(\alpha\gamma ac + \alpha\beta c + \delta c)v_{n-2l} + \alpha\gamma d}{(\gamma ac + \beta c)v_{n-2l} + \gamma d}, \quad n \geq l. \end{aligned} \quad (3.26)$$

Since both equations in (3.26) are solvable, we get the formulas

$$u_{2lm+i} = \begin{cases} \frac{A_{13}}{B_{13}} \frac{P_{131}\lambda_1^{m+1} - P_{132}\lambda_2^{m+1}}{P_{131}\lambda_1^m - P_{132}\lambda_2^m} - \frac{C_{13}}{B_{13}} & \text{if } R_{13} \neq \frac{1}{4}, \\ \frac{A_{13}}{B_{13}} \left( \frac{P_{13} + (P_{13} - A_{13})(m+1)}{2P_{13} + (2P_{13} - 2A_{13})m} \right) - \frac{C_{13}}{B_{13}} & \text{if } R_{13} = \frac{1}{4}, \end{cases} \quad (3.27)$$

and

$$v_{2lm+i} = \begin{cases} \frac{A_{13}}{B_{13}} \frac{\widehat{P}_{131}\lambda_1^{m+1} - \widehat{P}_{132}\lambda_2^{m+1}}{\widehat{P}_{131}\lambda_1^m - \widehat{P}_{132}\lambda_2^m} - \frac{\widehat{C}_{13}}{B_{13}} & \text{if } R_{13} \neq \frac{1}{4}, \\ \frac{A_{13}}{B_{13}} \left( \frac{\widehat{P}_{13} + (\widehat{P}_{13} - A_{13})(m+1)}{2\widehat{P}_{13} + (2\widehat{P}_{13} - 2A_{13})m} \right) - \frac{\widehat{C}_{13}}{B_{13}} & \text{if } R_{13} = \frac{1}{4}, \end{cases} \quad (3.28)$$

where  $R_{13} = \frac{d\gamma c\delta}{(aca\gamma + d\gamma + ca\beta + c\delta)^2}$ ,  $A_{13} = aca\gamma + d\gamma + ca\beta + c\delta$ ,  $B_{13} = ca\gamma$ ,  $C_{13} = ca\beta + c\delta$ ,  $\widehat{B}_{13} = \gamma ac + \beta c$ ,  $\widehat{C}_{13} = \gamma d$ ,  $P_{13j} = \lambda_j B_{13} \frac{x_{i-2l}}{y_{i-2l-k}} + \lambda_j C_{13} - R_{13} A_{13}$ ,  $\widehat{P}_{13j} = \lambda_j \widehat{B}_{13} \frac{y_{i-2l}}{x_{i-2l-k}} + \lambda_j \widehat{C}_{13} - R_{13} A_{13}$ ,  $P_{13} = 2B_{13} \frac{x_{i-2l}}{y_{i-2l-k}} + 2C_{13}$ ,  $\widehat{P}_{13} = 2\widehat{B}_{13} \frac{y_{i-2l}}{x_{i-2l-k}} + 2\widehat{C}_{13}$  for  $j \in \{1, 2\}$ ,  $i \in \{l, l+1, \dots, 3l-1\}$ . Putting (3.27) and (3.28) in (2.27) and (2.28), we get solutions of the system (1.5).

### 3.14. Case $\beta = 0$

This case is dual of the previous case. Hence, in this case, the system (1.5) is as follows:

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, \quad y_n = \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\gamma x_{n-l}}, \quad n \in \mathbb{N}_0. \quad (3.29)$$

By using (2.1), we get the system

$$u_n = \frac{acv_{n-l} + ab + d}{cv_{n-l} + b}, \quad v_n = \frac{\alpha\gamma u_{n-l} + \delta}{\gamma u_{n-l}}, \quad n \in \mathbb{N}_0, \quad (3.30)$$

from (3.29). Therefore, from the system (3.30), we get

$$\begin{aligned} u_n &= \frac{(aca\gamma + ab\gamma + d\gamma)u_{n-2l} + ac\delta}{(ca\gamma + b\gamma)u_{n-2l} + c\delta}, \\ v_n &= \frac{(aca\gamma + \delta c)v_{n-2l} + ab\alpha\gamma + \alpha\gamma d + \delta b}{\gamma acv_{n-2l} + \gamma ab + \gamma d}, \quad n \geq l. \end{aligned} \quad (3.31)$$

Since both equations of (3.31) are solvable, we get the formulas

$$u_{2lm+i} = \begin{cases} \frac{A_{14}}{B_{14}} \frac{P_{141}\lambda_1^{m+1} - P_{142}\lambda_2^{m+1}}{P_{141}\lambda_1^m - P_{142}\lambda_2^m} - \frac{C_{14}}{B_{14}} & \text{if } R_{14} \neq \frac{1}{4}, \\ \frac{A_{14}}{B_{14}} \left( \frac{P_{14} + (P_{14} - A_{14})(m+1)}{2P_{14} + (2P_{14} - 2A_{14})m} \right) - \frac{C_{14}}{B_{14}} & \text{if } R_{14} = \frac{1}{4} \end{cases} \quad (3.32)$$

$$v_{2lm+i} = \begin{cases} \frac{A_{14}}{B_{14}} \frac{\widehat{P}_{141}\lambda_1^{m+1} - \widehat{P}_{142}\lambda_2^{m+1}}{\widehat{P}_{141}\lambda_1^m - \widehat{P}_{142}\lambda_2^m} - \frac{\widehat{C}_{14}}{B_{14}} & \text{if } R_{14} \neq \frac{1}{4}, \\ \frac{A_{14}}{B_{14}} \left( \frac{\widehat{P}_{14} + (\widehat{P}_{14} - A_{14})(m+1)}{2\widehat{P}_{14} + (2\widehat{P}_{14} - 2A_{14})m} \right) - \frac{\widehat{C}_{14}}{B_{14}} & \text{if } R_{14} = \frac{1}{4}, \end{cases} \quad (3.33)$$

where  $R_{14} = \frac{d\gamma c\delta}{(aca\gamma + ab\gamma + d\gamma + c\delta)^2}$ ,  $A_{14} = aca\gamma + ab\gamma + d\gamma + c\delta$ ,  $B_{14} = ca\gamma + b\gamma$ ,  $C_{14} = c\delta$ ,  $\widehat{B}_{14} = \gamma ac$ ,  $\widehat{C}_{14} = \gamma ab + \gamma d$ ,  $P_{14j} = \lambda_j B_{14} \frac{x_{i-2l}}{y_{i-2l-k}} + \lambda_j C_{14} - R_{14} A_{14}$ ,  $\widehat{P}_{14j} = \lambda_j \widehat{B}_{14} \frac{y_{i-2l}}{x_{i-2l-k}} + \lambda_j \widehat{C}_{14} - R_{14} A_{14}$ ,  $P_{14} = 2B_{14} \frac{x_{i-2l}}{y_{i-2l-k}} + 2C_{14}$ ,  $\widehat{P}_{14} = 2\widehat{B}_{14} \frac{y_{i-2l}}{x_{i-2l-k}} + 2\widehat{C}_{14}$  for  $j \in \{1, 2\}$ ,  $i \in \{l, l+1, \dots, 3l-1\}$ . Putting (3.32) and (3.33) in (2.27) and (2.28), we get solutions of the system (1.5).

**3.15. Case  $b = \beta = 0$** 

If  $b = \beta = 0$  then the system (1.5) reduces to the following system

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{cy_{n-l}}, \quad y_n = \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\gamma x_{n-l}}, \quad n \in \mathbb{N}_0.$$

By using the changes of variables in (2.1) we get

$$u_n = \frac{acv_{n-l} + d}{cv_{n-l}}, \quad v_n = \frac{\alpha\gamma u_{n-l} + \delta}{\gamma u_{n-l}}, \quad n \in \mathbb{N}_0. \quad (3.34)$$

From the system (3.34) we get

$$u_n = \frac{(ac\alpha\gamma + \gamma d)u_{n-2l} + ac\delta}{c\alpha\gamma u_{n-2l} + c\delta}, \quad v_n = \frac{(\alpha\gamma ac + \delta c)v_{n-2l} + \alpha\gamma d}{\gamma acv_{n-2l} + \gamma d}, \quad n \geq l. \quad (3.35)$$

From (3.35) we get

$$u_{2lm+i} = \begin{cases} \frac{A_{15}}{B_{15}} \frac{P_{151}\lambda_1^{m+1} - (P_{152}\lambda_2^{m+1})}{P_{151}\lambda_1^m - P_{152}\lambda_2^m} - \frac{C_{15}}{B_{15}} & \text{if } R_{15} \neq \frac{1}{4}, \\ \frac{A_{15}}{B_{15}} \left( \frac{P_{15} + (P_{15} - A_{15})(m+1)}{2P_{15} + (2P_{15} - 2A_{15})m} \right) - \frac{C_{15}}{B_{15}} & \text{if } R_{15} = \frac{1}{4} \end{cases} \quad (3.36)$$

$$v_{2lm+i} = \begin{cases} \frac{\widehat{A}_{15}}{\widehat{B}_{15}} \frac{\widehat{P}_{151}\lambda_1^{m+1} - \widehat{P}_{152}\lambda_2^{m+1}}{\widehat{P}_{151}\lambda_1^m - \widehat{P}_{152}\lambda_2^m} - \frac{\widehat{C}_{15}}{\widehat{B}_{15}} & \text{if } R_{15} \neq \frac{1}{4}, \\ \frac{\widehat{A}_{15}}{\widehat{B}_{15}} \left( \frac{\widehat{P}_{15} + (\widehat{P}_{15} - A_{15})(m+1)}{2\widehat{P}_{15} + (2\widehat{P}_{15} - 2A_{15})m} \right) - \frac{\widehat{C}_{15}}{\widehat{B}_{15}} & \text{if } R_{15} = \frac{1}{4} \end{cases} \quad (3.37)$$

where  $R_{15} = \frac{cd\gamma\delta}{(ac\alpha\gamma + d\gamma + c\delta)^2}$ ,  $A_{15} = ac\alpha\gamma + \gamma d + c\delta$ ,  $B_{15} = c\alpha\gamma$ ,  $C_{15} = c\delta$ ,  $\widehat{B}_{15} = \gamma ac$ ,  $\widehat{C}_{15} = \gamma d$ ,  $P_{15j} = \lambda_j B_{15} \frac{x_{i-2l}}{y_{i-2l-k}} + \lambda_j C_{15} - R_{15} A_{15}$ ,  $\widehat{P}_{15j} = \lambda_j \widehat{B}_{15} \frac{y_{i-2l}}{x_{i-2l-k}} + \lambda_j \widehat{C}_{15} - R_{15} A_{15}$ ,  $P_{15} = 2B_{15} \frac{x_{i-2l}}{y_{i-2l-k}} + 2C_{15}$ ,  $\widehat{P}_{15} = 2\widehat{B}_{15} \frac{y_{i-2l}}{x_{i-2l-k}} + 2\widehat{C}_{15}$  for  $j \in \{1, 2\}$ ,  $i \in \{l, l+1, \dots, 3l-1\}$ . Putting (3.36) and (3.37) in (2.27) and (2.28), we get solutions of the system (1.5).

**3.16. Case  $abcd \neq 0$** 

Here we deal with the case when  $abcd \neq 0$ . Since in this case the system (1.5) can be written in the form of

$$x_n = ay_{n-k} + \frac{y_{n-k}x_{n-(k+l)}}{b_1x_{n-(k+l)} + c_1y_{n-l}}, \quad y_n = \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0,$$

with  $b_1 = \frac{b}{d}$  and  $c_1 = \frac{c}{d}$ , we may assume that  $d = 1$ . Hence we will consider the system

$$\begin{aligned} x_n &= ay_{n-k} + \frac{y_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, \\ y_n &= \alpha x_{n-k} + \frac{\delta x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0, \end{aligned} \quad (3.38)$$

from now on. Moreover, the system (3.38) can be written in the form of

$$\begin{aligned} \frac{x_n}{y_{n-k}} &= \frac{(ab\gamma + \gamma + ac\alpha\gamma) \frac{x_{n-2l}}{y_{n-(k+2l)}} + ab\beta + \beta + ac\alpha\beta + ac\delta}{(b\gamma + c\alpha\gamma) \frac{x_{n-2l}}{y_{n-(k+2l)}} + b\beta + c\alpha\beta + c\delta}, \\ \frac{y_n}{x_{n-k}} &= \frac{\alpha\gamma \frac{x_{n-l}}{y_{n-(k+l)}} + (\alpha\beta + \delta)}{\gamma \frac{x_{n-l}}{y_{n-(k+l)}} + \beta}, \quad n \geq l. \end{aligned} \quad (3.39)$$

**Remark 3.1.** For  $abca\beta\gamma\delta \neq 0$  in the system (1.5), it is easy to see that there is the degenerate case

$$\begin{vmatrix} aca\gamma + ab\gamma + d\gamma & aca\beta + ac\delta + ab\beta + d\beta \\ ca\gamma + b\gamma & ca\beta + c\delta + b\beta \end{vmatrix} = 0$$

if and only if  $d = 0$ . Hence, we avoid the degenerate case via the assumption  $d \neq 0$ .

**3.16.1. The case  $ab\gamma + \gamma + aca\gamma + b\beta + ca\beta + c\delta = 0$ .** If  $ab\gamma + \gamma + aca\gamma + b\beta + ca\beta + c\delta = 0$ , then we get the system

$$\begin{aligned} \frac{x_n}{y_{n-k}} &= \frac{-(b\beta + ca\beta + c\delta) \frac{x_{n-2l}}{y_{n-(k+2l)}} + \frac{-\beta(b\beta + ca\beta + c\delta) + ac\gamma\delta}{\gamma}}{(b\gamma + ca\gamma) \frac{x_{n-2l}}{y_{n-(k+2l)}} + b\beta + ca\beta + c\delta}, \\ \frac{y_n}{x_{n-k}} &= \frac{\alpha\gamma \frac{x_{n-l}}{y_{n-(k+l)}} + (\alpha\beta + \delta)}{\gamma \frac{x_{n-l}}{y_{n-(k+l)}} + \beta}, \quad n \geq l, \end{aligned} \quad (3.40)$$

from (3.39). By using the change of variables (2.1), we get the system

$$\begin{aligned} u_n &= \frac{-(b\beta + ca\beta + c\delta) u_{n-2l} + \frac{-\beta(b\beta + ca\beta + c\delta) + ac\gamma\delta}{\gamma}}{(b\gamma + ca\gamma) u_{n-2l} + b\beta + ca\beta + c\delta}, \\ v_n &= \frac{\alpha\gamma u_{n-l} + (\alpha\beta + \delta)}{\gamma u_{n-l} + \beta}, \quad n \geq l, \end{aligned} \quad (3.41)$$

which can be written as

$$\begin{aligned} (b\gamma + ca\gamma) u_n + b\beta + ca\beta + c\delta &= \frac{abc\gamma\delta + ac^2\alpha\gamma\delta + c^2\alpha\beta\delta + c\delta b\beta + c^2\delta^2}{(b\gamma + ca\gamma) u_{n-2l} + b\beta + ca\beta + c\delta}, \\ v_n &= \frac{\alpha\gamma u_{n-l} + (\alpha\beta + \delta)}{\gamma u_{n-l} + \beta}, \quad n \geq l. \end{aligned} \quad (3.42)$$

By applying the change of variables  $(b\gamma + ca\gamma) u_n + b\beta + ca\beta + c\delta = t_n$  to the system (3.42), we obtain

$$\begin{aligned} t_n &= \frac{abc\gamma\delta + ac^2\alpha\gamma\delta + c^2\alpha\beta\delta + c\delta b\beta + c^2\delta^2}{t_{n-2l}} \\ &= t_{n-4l} \\ &= c_j, \quad n \geq 3l, \quad j = \overline{1, 4l}, \end{aligned} \quad (3.43)$$

where each  $c_j$  is a constant which depends to the initial values  $x_{-i}, y_{-i}, i = \overline{1, k+l}$ . Consequently, by using  $(b\gamma + ca\gamma) u_n + b\beta + ca\beta + c\delta = t_n$  and considering the system (3.42), we get

$$u_n = \frac{t_n - (b\beta + ca\beta + c\delta)}{b\gamma + ca\gamma}, \quad n \geq -l, \quad v_n = \frac{\alpha t_{n-l} + \delta b}{t_{n-l} - c\delta}, \quad n \in \mathbb{N}_0. \quad (3.44)$$

Putting (3.44) in (2.27) and (2.28), we get solutions of the system (1.5).

### 3.17. Case $\alpha\beta\gamma\delta \neq 0$

Here we deal with the case when  $\alpha\beta\gamma\delta \neq 0$ . Since in this case the system (1.5) can be written in the form of

$$x_n = ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, \quad y_n = \alpha x_{n-k} + \frac{x_{n-k}y_{n-(k+l)}}{\beta_1 y_{n-(k+l)} + \gamma_1 x_{n-l}}, \quad n \in \mathbb{N}_0,$$

with  $\beta_1 = \frac{\beta}{\delta}$  and  $\gamma_1 = \frac{\gamma}{\delta}$ , we may assume that  $\delta = 1$ . Hence we will consider the system

$$\begin{aligned}x_n &= ay_{n-k} + \frac{dy_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, \\y_n &= \alpha x_{n-k} + \frac{x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, \quad n \in \mathbb{N}_0,\end{aligned}\tag{3.45}$$

from now on. Moreover, the system (3.45) can be written in the form of

$$\begin{aligned}\frac{x_n}{y_{n-k}} &= \frac{ac \frac{y_{n-l}}{x_{n-(k+l)}} + (ab+d)}{c \frac{y_{n-l}}{x_{n-(k+l)}} + b}, \\ \frac{y_n}{x_{n-k}} &= \frac{(\alpha\beta c + c + \alpha\gamma ac) \frac{y_{n-2l}}{x_{n-(k+2l)}} + \alpha\beta b + b + \alpha\gamma ab + \alpha\gamma d}{(\beta c + \gamma ac) \frac{y_{n-2l}}{x_{n-(k+2l)}} + \beta b + \gamma ab + \gamma d}, \quad n \geq l.\end{aligned}\tag{3.46}$$

**Remark 3.2.** For  $\alpha\beta\gamma abcd \neq 0$  in the system (1.5), it is easy to see that there is the degenerate case

$$\begin{vmatrix} \alpha\gamma ac + \alpha\beta c + \delta c & \alpha\gamma ab + \alpha\gamma d + \alpha\beta b + \delta b \\ \gamma ac + \beta c & \gamma ab + \gamma d + \beta b \end{vmatrix} = 0$$

if and only if  $\delta = 0$ . Hence, we avoid the degenerate case via the assumption  $\delta \neq 0$ .

**3.17.1. The case  $\alpha\beta c + c + \alpha\gamma ac + \beta b + \gamma ab + \gamma d = 0$ .** If  $\alpha\beta c + c + \alpha\gamma ac + \beta b + \gamma ab + \gamma d = 0$ , then we get the system

$$\begin{aligned}\frac{x_n}{y_{n-k}} &= \frac{ac \frac{y_{n-l}}{x_{n-(k+l)}} + (ab+d)}{c \frac{y_{n-l}}{x_{n-(k+l)}} + b}, \\ \frac{y_n}{x_{n-k}} &= \frac{-(\beta b + \gamma ab + \gamma d) \frac{y_{n-2l}}{x_{n-(k+2l)}} + \frac{-b(\beta b + \gamma ab + \gamma d) + \alpha\gamma cd}{c}}{(\beta c + \gamma ac) \frac{y_{n-2l}}{x_{n-(k+2l)}} + \beta b + \gamma ab + \gamma d}, \quad n \geq l,\end{aligned}\tag{3.47}$$

from (3.46). By using the change of variables (2.1), we get the system

$$\begin{aligned}u_n &= \frac{acv_{n-l} + (ab+d)}{cv_{n-l} + b}, \\ v_n &= \frac{-(\beta b + \gamma ab + \gamma d)v_{n-2l} + \frac{-b(\beta b + \gamma ab + \gamma d) + \alpha\gamma cd}{c}}{(\beta c + \gamma ac)v_{n-2l} + \beta b + \gamma ab + \gamma d}, \quad n \geq l,\end{aligned}\tag{3.48}$$

which can be written as

$$\begin{aligned}u_n &= \frac{acv_{n-l} + (ab+d)}{cv_{n-l} + b}, \quad n \geq l, \\ (\beta c + \gamma ac)v_n + \beta b + \gamma ab + \gamma d &= \frac{\alpha\beta\gamma cd + \alpha\gamma^2 acd + \gamma^2 abd + \gamma d\beta b + \gamma^2 d^2}{(\beta c + \gamma ac)v_{n-2l} + \beta b + \gamma ab + \gamma d}, \quad n \geq l.\end{aligned}\tag{3.49}$$

Next, by applying the change of variables  $(\beta c + \gamma ac)v_n + \beta b + \gamma ab + \gamma d = \hat{t}_n$  to the system (3.49), we obtain

$$\begin{aligned}\hat{t}_n &= \frac{\alpha\beta\gamma cd + \alpha\gamma^2 acd + \gamma^2 abd + \gamma d\beta b + \gamma^2 d^2}{\hat{t}_{n-2l}} \\ &= \hat{t}_{n-4l} \\ &= d_j, \quad n \geq 3l, \quad j = \overline{1, 4l},\end{aligned}\tag{3.50}$$

where each  $d_j$  is a constant which depends to the initial values  $x_{-i}, y_{-i}, i = \overline{1, k+l}$ . Consequently, by using  $(\beta c + \gamma ac)v_n + \beta b + \gamma ab + \gamma d = \hat{t}_n$  and considering the system

(3.49), we get

$$u_n = \frac{\widehat{at}_{n-l} + d\beta}{\widehat{t}_{n-l} - \gamma d}, n \in \mathbb{N}_0, v_n = \frac{\widehat{t}_n - (\beta b + \gamma ab + \gamma d)}{\beta c + \gamma ac}, n \geq -l. \quad (3.51)$$

Putting (3.51) in (2.27) and (2.28), we get solutions of the system (1.5).

### 3.18. Case $abcd\alpha\beta\gamma\delta \neq 0$

Here we deal with the case when  $abcd\alpha\beta\gamma\delta \neq 0$ . Since in this case the system (1.5) can be written in the form of

$$x_n = ay_{n-k} + \frac{y_{n-k}x_{n-(k+l)}}{b_2x_{n-(k+l)} + c_2y_{n-l}}, y_n = \alpha x_{n-k} + \frac{x_{n-k}y_{n-(k+l)}}{\beta_2y_{n-(k+l)} + \gamma_2x_{n-l}}, n \in \mathbb{N}_0,$$

with  $b_2 = \frac{b}{d}$ ,  $c_2 = \frac{c}{d}$ ,  $\beta_2 = \frac{\beta}{\delta}$  and  $\gamma_2 = \frac{\gamma}{\delta}$ , we may assume that  $d = 1$  and  $\delta = 1$ . Hence we will consider the system

$$\begin{aligned} x_n &= ay_{n-k} + \frac{y_{n-k}x_{n-(k+l)}}{bx_{n-(k+l)} + cy_{n-l}}, \\ y_n &= \alpha x_{n-k} + \frac{x_{n-k}y_{n-(k+l)}}{\beta y_{n-(k+l)} + \gamma x_{n-l}}, n \in \mathbb{N}_0, \end{aligned} \quad (3.52)$$

from now on. Moreover, the system (3.52) can be written in the form of

$$\begin{aligned} \frac{x_n}{y_{n-k}} &= \frac{(ab\gamma + aca\gamma + \gamma) \frac{x_{n-2l}}{y_{n-(k+2l)}} + ab\beta + \beta + aca\beta + ac}{(b\gamma + ca\gamma) \frac{x_{n-2l}}{y_{n-(k+2l)}} + b\beta + ca\beta + c}, n \geq l, \\ \frac{y_n}{x_{n-k}} &= \frac{(\alpha\beta c + \alpha\gamma ac + c) \frac{y_{n-2l}}{x_{n-(k+2l)}} + \alpha\beta b + b + \alpha\gamma ab + \alpha\gamma}{(\beta c + \gamma ac) \frac{y_{n-2l}}{x_{n-(k+2l)}} + \beta b + \gamma ab + \gamma}, n \geq l. \end{aligned} \quad (3.53)$$

**Remark 3.3.** For  $abca\beta\gamma \neq 0$  in the system (1.5), it is easy to see that there is the degenerate case

$$\begin{aligned} \left| \begin{array}{cc} aca\gamma + ab\gamma + d\gamma & aca\beta + ac\delta + ab\beta + d\beta \\ ca\gamma + b\gamma & ca\beta + c\delta + b\beta \end{array} \right| &= 0, \\ \left| \begin{array}{cc} \alpha\gamma ac + \alpha\beta c + \delta c & \alpha\gamma ab + \alpha\gamma d + \alpha\beta b + \delta b \\ \gamma ac + \beta c & \gamma ab + \gamma d + \beta b \end{array} \right| &= 0, \end{aligned}$$

if and only if  $d = 0$  and  $\delta = 0$ . Hence, we avoid the degenerate case via the assumptions  $d \neq 0$  and  $\delta \neq 0$ .

**3.18.1. The case  $ab\gamma + aca\gamma + b\beta + ca\beta + c + \gamma = 0$ .** If  $ab\gamma + aca\gamma + b\beta + ca\beta + c + \gamma = 0$ , then we get the system

$$\begin{aligned} \frac{x_n}{y_{n-k}} &= \frac{-(b\beta + ca\beta + c) \frac{x_{n-2l}}{y_{n-(k+2l)}} + \frac{-\beta(b\beta + ca\beta + c) + ac\gamma}{\gamma}}{(b\gamma + ca\gamma) \frac{x_{n-2l}}{y_{n-(k+2l)}} + b\beta + ca\beta + c}, n \geq l, \\ \frac{y_n}{x_{n-k}} &= \frac{-(\beta b + \gamma ab + \gamma) \frac{y_{n-2l}}{x_{n-(k+2l)}} + \frac{-b(\beta b + \gamma ab + \gamma) + \alpha\gamma c}{c}}{(\beta c + \gamma ac) \frac{y_{n-2l}}{x_{n-(k+2l)}} + \beta b + \gamma ab + \gamma}, n \geq l, \end{aligned} \quad (3.54)$$

from (3.53). By using the change of variables (2.1), we get the system

$$\begin{aligned} u_n &= \frac{-(b\beta + ca\beta + c) u_{n-2l} + \frac{-\beta(b\beta + ca\beta + c) + ac\gamma}{\gamma}}{(b\gamma + ca\gamma) u_{n-2l} + b\beta + ca\beta + c}, n \geq l, \\ v_n &= \frac{-(\beta b + \gamma ab + \gamma) v_{n-2l} + \frac{-b(\beta b + \gamma ab + \gamma) + \alpha\gamma c}{c}}{(\beta c + \gamma ac) v_{n-2l} + \beta b + \gamma ab + \gamma}, n \geq l, \end{aligned} \quad (3.55)$$

which can be written as

$$\begin{aligned} (b\gamma + c\alpha\gamma)u_n + b\beta + c\alpha\beta + c &= \frac{abc\gamma + ac^2\alpha\gamma + cb\beta + c^2\alpha\beta + c^2}{(b\gamma + c\alpha\gamma)u_{n-2l} + b\beta + c\alpha\beta + c}, n \geq l, \\ (\beta c + \gamma ac)v_n + \beta b + \gamma ab + \gamma &= \frac{\alpha\beta\gamma c + \alpha\gamma^2 ac + \gamma\beta b + \gamma^2 ab + \gamma^2}{(\beta c + \gamma ac)v_{n-2l} + \beta b + \gamma ab + \gamma}, n \geq l. \end{aligned} \quad (3.56)$$

Next, by applying the change of variables  $(b\gamma + c\alpha\gamma)u_n + b\beta + c\alpha\beta + c = k_n$  and  $(\beta c + \gamma ac)v_n + \beta b + \gamma ab + \gamma = \widehat{k}_n$  to the system (3.56), we obtain

$$\begin{aligned} k_n &= \frac{abc\gamma + ac^2\alpha\gamma + cb\beta + c^2\alpha\beta + c^2}{k_{n-2l}} = k_{n-4l} = e_j, n \geq 3l, j = \overline{1, 4l}, \\ \widehat{k}_n &= \frac{\alpha\beta\gamma c + \alpha\gamma^2 ac + \gamma\beta b + \gamma^2 ab + \gamma^2}{\widehat{k}_{n-2l}} = \widehat{k}_{n-4l} = \widehat{e}_j, n \geq 3l, j = \overline{1, 4l}, \end{aligned} \quad (3.57)$$

where each  $e_j$  and  $\widehat{e}_j$  are constants which dependent to the initial values  $x_{-i}, y_{-i}, i = \overline{1, k+l}$ . Consequently, by using  $(b\gamma + c\alpha\gamma)u_n + b\beta + c\alpha\beta + c = k_n$  and  $(\beta c + \gamma ac)v_n + \beta b + \gamma ab + \gamma = \widehat{k}_n$ , we get

$$u_n = \frac{k_n - (b\beta + c\alpha\beta + c)}{b\gamma + c\alpha\gamma}, v_n = \frac{\widehat{k}_n - (\beta b + \gamma ab + \gamma)}{\beta c + \gamma ac}, n \geq -l. \quad (3.58)$$

Putting (3.58) in (2.27) and (2.28), we get solutions of the system (1.5).

#### 4. Long-term behavior of the system for $abc\alpha\beta\gamma \neq 0$ when $l = 1$

In this section, we investigate the long-term behavior of the solutions of the system (1.5) when  $l = 1, abc\alpha\beta\gamma \neq 0$ . In this case, we get the system

$$x_n = ay_{n-k} + \frac{y_{n-k}x_{n-k-1}}{bx_{n-k-1} + cy_{n-1}}, y_n = \alpha x_{n-k} + \frac{x_{n-k}y_{n-k-1}}{\beta y_{n-k-1} + \gamma x_{n-1}}, n \in \mathbb{N}_0. \quad (4.1)$$

The solutions of the system (4.1) are given by

$$x_{2kn+2r+i} = \begin{cases} x_{2r-2k+i} \prod_{j=0}^n \left( \frac{D}{E_1} \frac{N_1 \lambda_1^{kj+r+1} - N_2 \lambda_2^{kj+r+1}}{N_1 \lambda_1^{kj+r} - N_2 \lambda_2^{kj+r}} - \frac{F_1}{E_1} \right) \\ \quad \times \left( \frac{D}{E_2} \frac{T_1 \lambda_1^{kj+r+\lfloor \frac{i-k-1}{2} \rfloor + 1} - T_2 \lambda_2^{kj+r+\lfloor \frac{i-k-1}{2} \rfloor + 1}}{T_1 \lambda_1^{kj+r+\lfloor \frac{i-k-1}{2} \rfloor} - T_2 \lambda_2^{kj+r+\lfloor \frac{i-k-1}{2} \rfloor}} - \frac{F_2}{E_2} \right) & \text{if } \widetilde{R} < \frac{1}{4}, \\ x_{2r-2k+i} \prod_{j=0}^n \left( \frac{D}{E_1} \left( \frac{Z_1 + (Z_1 - D)(kj+r+1)}{2Z_1 + (2Z_1 - 2D)(kj+r)} \right) - \frac{F_1}{E_1} \right) \\ \quad \times \left( \frac{D}{E_2} \left( \frac{Z_2 + (Z_2 - D)(kj+r+\lfloor \frac{i-k-1}{2} \rfloor + 1)}{2Z_2 + (2Z_2 - 2D)(kj+r+\lfloor \frac{i-k-1}{2} \rfloor)} \right) - \frac{F_2}{E_2} \right) & \text{if } \widetilde{R} = \frac{1}{4}, \end{cases} \quad (4.2)$$

and

$$y_{2kn+2r+i} = \begin{cases} y_{2r-2k+i} \prod_{j=0}^n \left( \frac{D}{E_2} \frac{T_1 \lambda_1^{kj+r+1} - T_2 \lambda_2^{kj+r+1}}{T_1 \lambda_1^{kj+r} - T_2 \lambda_2^{kj+r}} - \frac{F_2}{E_2} \right) \\ \quad \times \left( \frac{D}{E_1} \frac{N_1 \lambda_1^{kj+r+\lfloor \frac{i-k-1}{2} \rfloor + 1} - N_2 \lambda_2^{kj+r+\lfloor \frac{i-k-1}{2} \rfloor + 1}}{N_1 \lambda_1^{kj+r+\lfloor \frac{i-k-1}{2} \rfloor} - N_2 \lambda_2^{kj+r+\lfloor \frac{i-k-1}{2} \rfloor}} - \frac{F_1}{E_1} \right) & \text{if } \widetilde{R} < \frac{1}{4}, \\ y_{2r-2k+i} \prod_{j=0}^n \left( \frac{D}{E_2} \left( \frac{Z_2 + (Z_2 - D)(kj+r+1)}{2Z_2 + (2Z_2 - 2D)(kj+r)} \right) - \frac{F_2}{E_2} \right) \\ \quad \times \left( \frac{D}{E_1} \left( \frac{Z_1 + (Z_1 - D)(kj+r+\lfloor \frac{i-k-1}{2} \rfloor + 1)}{2Z_1 + (2Z_1 - 2D)(kj+r+\lfloor \frac{i-k-1}{2} \rfloor)} \right) - \frac{F_1}{E_1} \right) & \text{if } \widetilde{R} = \frac{1}{4}, \end{cases} \quad (4.3)$$

where  $\widetilde{R} = \frac{c\gamma}{(aca\gamma + ab\gamma + \gamma + c\alpha\beta + c + b\beta)^2}$ ,  $D = aca\gamma + ab\gamma + c\alpha\beta + b\beta + c + \gamma$ ,  $E_1 = c\alpha\gamma + b\gamma$ ,  $F_1 = c\alpha\beta + c + b\beta$ ,  $E_2 = \gamma ac + \beta c$ ,  $F_2 = \gamma ab + \gamma + \beta b$ ,  $Z_1 = 2E_1 \frac{x_{i-2}}{y_{i-k-2}} + 2F_1$ ,  $Z_2 = 2E_2 \frac{y_{i-2}}{x_{i-k-2}} + 2F_2$ ,  $N_j := \lambda_j E_1 \frac{x_{i-2}}{y_{i-k-2}} + \lambda_j F_1 - \widetilde{R}D$  and  $T_j := \lambda_j E_2 \frac{y_{i-2}}{x_{i-k-2}} + \lambda_j F_2 - \widetilde{R}D$  for



$i \in \{1, 2\}, j \in \{1, 2\}, k \in \mathbb{N}$  and  $r \in \mathbb{S}_{k,i}$ , where

$$\mathbb{S}_{k,i} = \begin{cases} \left\{ \left\lfloor \frac{k-i-1}{2} \right\rfloor + i_1; i_1 = \overline{0, k-1} \right\} & \text{if } (k-i) \text{ odd,} \\ \left\{ \left\lfloor \frac{k-i-1}{2} \right\rfloor + i_1; i_1 = \overline{1, k} \right\} & \text{if } (k-i) \text{ even.} \end{cases}$$

**Theorem 4.1.** Assume that  $\tilde{R} = \frac{c\gamma}{(aca\gamma+ab\gamma+\gamma+c\alpha\beta+c+b\beta)^2} < \frac{1}{4}, \frac{x_{i-2}}{y_{i-k-2}} \neq \frac{\tilde{R}D-\lambda_j F_1}{\lambda_j E_1}, \frac{y_{i-2}}{x_{i-k-2}} \neq \frac{\tilde{R}D-\lambda_j F_2}{\lambda_j E_2}, L_j := \frac{D\lambda_j-F_1}{E_1}, M_j := \frac{D\lambda_j-F_2}{E_2}, N_j := \lambda_j E_1 \frac{x_{i-2}}{y_{i-k-2}} + \lambda_j F_1 - \tilde{R}D$  and  $T_j := \lambda_j E_2 \frac{y_{i-2}}{x_{i-k-2}} + \lambda_j F_2 - \tilde{R}D$  for  $i \in \{1, 2\}$  and  $j \in \{1, 2\}$ . Then the following statements are true.

- (a) If  $|\lambda_1| > |\lambda_2|$  and  $|L_1 M_1| < 1$ , then  $x_n \rightarrow 0$  and  $y_n \rightarrow 0$ , as  $n \rightarrow \infty$ .
- (b) If  $|\lambda_1| > |\lambda_2|$  and  $|L_1 M_1| > 1$ , then  $|x_n| \rightarrow \infty$  and  $|y_n| \rightarrow \infty$ , as  $n \rightarrow \infty$ .
- (c) If  $|\lambda_1| > |\lambda_2|$  and  $L_1 M_1 = 1$ , then  $(x_n)_{n \geq -k-1}$  and  $(y_n)_{n \geq -k-1}$  are convergent.
- (d) If  $|\lambda_1| > |\lambda_2|$  and  $L_1 M_1 = -1$ , then  $(x_{2kn+2r+i})_{n \in \mathbb{N}_0}$  and  $(y_{2kn+2r+i})_{n \in \mathbb{N}_0}$ , for  $i \in \{1, 2\}, r \in \mathbb{S}_{k,i}$  are convergent.
- (e) If  $|\lambda_1| < |\lambda_2|$  and  $|L_2 M_2| < 1$ , then  $x_n \rightarrow 0$  and  $y_n \rightarrow 0$ , as  $n \rightarrow \infty$ .
- (f) If  $|\lambda_1| < |\lambda_2|$  and  $|L_2 M_2| > 1$ , then  $|x_n| \rightarrow \infty$  and  $|y_n| \rightarrow \infty$ , as  $n \rightarrow \infty$ .
- (g) If  $|\lambda_1| < |\lambda_2|$  and  $L_2 M_2 = 1$ , then  $(x_n)_{n \geq -k-1}$  and  $(y_n)_{n \geq -k-1}$  are convergent.
- (h) If  $|\lambda_1| < |\lambda_2|$  and  $L_2 M_2 = -1$ , then  $(x_{2kn+2r+i})_{n \in \mathbb{N}_0}$  and  $(y_{2kn+2r+i})_{n \in \mathbb{N}_0}$ , for  $i \in \{1, 2\}, r \in \mathbb{S}_{k,i}$  are convergent.

**Proof.** Suppose that

$$\begin{aligned} a_{n_1}^{(r)} &= \left( \frac{D}{E_1} \frac{N_1 \lambda_1^{kn_1+r+1} - N_2 \lambda_2^{kn_1+r+1}}{N_1 \lambda_1^{kn_1+r} - N_2 \lambda_2^{kn_1+r}} - \frac{F_1}{E_1} \right) \\ &\times \left( \frac{D}{E_2} \frac{T_1 \lambda_1^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor + 1} - T_2 \lambda_2^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor + 1}}{T_1 \lambda_1^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} - T_2 \lambda_2^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor}} - \frac{F_2}{E_2} \right) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} b_{n_1}^{(r)} &= \left( \frac{D}{E_2} \frac{T_1 \lambda_1^{kn_1+r+1} - T_2 \lambda_2^{kn_1+r+1}}{T_1 \lambda_1^{kn_1+r} - T_2 \lambda_2^{kn_1+r}} - \frac{F_2}{E_2} \right) \\ &\times \left( \frac{D}{E_1} \frac{N_1 \lambda_1^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor + 1} - N_2 \lambda_2^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor + 1}}{N_1 \lambda_1^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} - N_2 \lambda_2^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor}} - \frac{F_1}{E_1} \right) \end{aligned} \tag{4.5}$$

for  $n_1 \in \mathbb{N}_0, i \in \{1, 2\}$  and  $r \in \mathbb{S}_{k,i}$ . Then, if  $|\lambda_1| > |\lambda_2|$ , we get

$$\lim_{n_1 \rightarrow \infty} a_{n_1}^{(r)} = \lim_{n_1 \rightarrow \infty} b_{n_1}^{(r)} = \left( \frac{D\lambda_1 - F_1}{E_1} \right) \left( \frac{D\lambda_1 - F_2}{E_2} \right) \tag{4.6}$$

for each  $i \in \{1, 2\}$ . From (4.2), (4.3), and (4.6), the results follow from the assumptions in (a) and (b). For each  $i \in \{1, 2\}$  and sufficiently large  $n_1$ , (4.4) and (4.5) can be written as

$$\begin{aligned}
 a_{n_1}^{(r)} &= \left( -\frac{F_1}{E_1} + \frac{D}{E_1} \frac{\lambda_1 \left( 1 - \frac{N_2}{N_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+1} \right)}{1 - \frac{N_2}{N_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r}} \right) \\
 &\quad \times \left( -\frac{F_2}{E_2} + \frac{D}{E_2} \frac{\lambda_1 \left( 1 - \frac{T_2}{T_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor + 1} \right)}{1 - \frac{T_2}{T_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor}} \right) \\
 &= \left( -\frac{F_1}{E_1} + \frac{D\lambda_1}{E_1} + \frac{DN_2}{E_1N_1} (\lambda_1 - \lambda_2) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right) \\
 &\quad \times \left( -\frac{F_2}{E_2} + \frac{D\lambda_1}{E_2} + \frac{DT_2}{E_2T_1} (\lambda_1 - \lambda_2) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right) \\
 &= L_1M_1 + L_1 \frac{DT_2}{E_2T_1} (\lambda_1 - \lambda_2) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} \\
 &\quad + M_1 \frac{DN_2}{E_1N_1} (\lambda_1 - \lambda_2) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1}
 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
 b_{n_1}^{(r)} &= \left( -\frac{F_2}{E_2} + \frac{D}{E_2} \frac{\lambda_1 \left( 1 - \frac{T_2}{T_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+1} \right)}{1 - \frac{T_2}{T_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r}} \right) \\
 &\quad \times \left( -\frac{F_1}{E_1} + \frac{D}{E_1} \frac{\lambda_1 \left( 1 - \frac{N_2}{N_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor + 1} \right)}{1 - \frac{N_2}{N_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor}} \right) \\
 &= \left( -\frac{F_2}{E_2} + \frac{D\lambda_1}{E_2} + \frac{DT_2}{E_2T_1} (\lambda_1 - \lambda_2) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right) \\
 &\quad \times \left( -\frac{F_1}{E_1} + \frac{D\lambda_1}{E_1} + \frac{DN_2}{E_1N_1} (\lambda_1 - \lambda_2) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right) \\
 &= M_1L_1 + M_1 \frac{DN_2}{E_1N_1} (\lambda_1 - \lambda_2) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} \\
 &\quad + L_1 \frac{DT_2}{E_2T_1} (\lambda_1 - \lambda_2) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1}.
 \end{aligned} \tag{4.8}$$

From (4.2), (4.3), (4.7), and (4.8), the results in (c) can be seen easily. For each  $i \in \{1, 2\}$  and sufficiently large  $n_1$ , (4.4) and (4.5) can be again written as

$$\begin{aligned}
 a_{n_1}^{(r)} &= \left( -\frac{F_1}{E_1} + \frac{D}{E_1} \frac{-\lambda_1 \left( -1 + \frac{N_2}{N_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+1} \right)}{1 - \frac{N_2}{N_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r}} \right) \\
 &\quad \times \left( -\frac{F_2}{E_2} + \frac{D}{E_2} \frac{-\lambda_1 \left( -1 + \frac{T_2}{T_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor + 1} \right)}{1 - \frac{T_2}{T_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor}} \right) \\
 &= \left( -\frac{F_1}{E_1} + \frac{D\lambda_1}{E_1} \left( 1 - \frac{N_2(\lambda_2 - \lambda_1)}{N_1 \lambda_1} \right) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right) \\
 &\quad \times \left( -\frac{F_2}{E_2} + \frac{D\lambda_1}{E_2} \left( 1 - \frac{T_2(\lambda_2 - \lambda_1)}{T_1 \lambda_1} \right) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right) \\
 &= L_1 M_1 \left( 1 - \frac{N_2(\lambda_2 - \lambda_1)}{N_1 \lambda_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r} - \frac{T_2(\lambda_2 - \lambda_1)}{T_1 \lambda_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} \right. \\
 &\quad \left. + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right)
 \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
 b_{n_1}^{(r)} &= \left( -\frac{F_2}{E_2} + \frac{D}{E_2} \frac{-\lambda_1 \left( -1 + \frac{T_2}{T_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+1} \right)}{1 - \frac{T_2}{T_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r}} \right) \\
 &\quad \times \left( -\frac{F_1}{E_1} + \frac{D}{E_1} \frac{-\lambda_1 \left( -1 + \frac{N_2}{N_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor + 1} \right)}{1 - \frac{N_2}{N_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor}} \right) \\
 &= \left( -\frac{F_2}{E_2} + \frac{D\lambda_1}{E_2} \left( 1 - \frac{T_2(\lambda_2 - \lambda_1)}{T_1 \lambda_1} \right) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right) \\
 &\quad \times \left( -\frac{F_1}{E_1} + \frac{D\lambda_1}{E_1} \left( 1 - \frac{N_2(\lambda_2 - \lambda_1)}{N_1 \lambda_1} \right) \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right) \\
 &= M_1 L_1 \left( 1 - \frac{T_2(\lambda_2 - \lambda_1)}{T_1 \lambda_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r} - \frac{N_2(\lambda_2 - \lambda_1)}{N_1 \lambda_1} \left( \frac{\lambda_2}{\lambda_1} \right)^{kn_1+r+\lfloor \frac{i-k-1}{2} \rfloor} \right. \\
 &\quad \left. + \mathcal{O} \left( \frac{\lambda_2}{\lambda_1} \right)^{2kn_1} \right)
 \end{aligned} \tag{4.10}$$

From (4.2), (4.3), (4.9), and (4.10), the results in (d) can be seen easily. Proofs of the (e)-(h) are not given in here since they could be obtained similar with proofs of the (a)-(d).  $\square$

**Theorem 4.2.** Assume that  $\tilde{R} = \frac{c\gamma}{(aca\gamma+ab\gamma+\gamma+c\alpha\beta+c+b\beta)^2} = \frac{1}{4}$ ,  $x_{i-k-2} y_{i-k-2} \neq 0$  for  $i \in \{1, 2\}$ ,  $D = aca\gamma+ab\gamma+c\alpha\beta+b\beta+c+\gamma$ ,  $Z_1 = 2E_1 \frac{x_{i-2}}{y_{i-k-2}} + 2F_1$ ,  $Z_2 = 2E_2 \frac{y_{i-2}}{x_{i-k-2}} + 2F_2$ . Then the following statements are true.

- (a) If  $|\frac{(D-2F_1)(D-2F_2)}{4E_1E_2}| < 1$ , then  $x_n \rightarrow 0$  and  $y_n \rightarrow 0$ , as  $n \rightarrow \infty$ .

- (b) If  $|\frac{(D-2F_1)(D-2F_2)}{4E_1E_2}| > 1$ , then  $|x_n| \rightarrow \infty$  and  $|y_n| \rightarrow \infty$ , as  $n \rightarrow \infty$ .  
 (c) If  $|\frac{(D-2F_1)(D-2F_2)}{4E_1E_2}| = 1$  and  $\frac{(D-2F_1)(D-2F_2)}{2D(D-F_1-F_2)} > 0$ , then  $|x_n| \rightarrow \infty$  and  $|y_n| \rightarrow \infty$ , as  $n \rightarrow \infty$ .  
 (d) If  $|\frac{(D-2F_1)(D-2F_2)}{4E_1E_2}| = 1$  and  $\frac{(D-2F_1)(D-2F_2)}{2D(D-F_1-F_2)} < 0$ , then  $x_n \rightarrow 0$  and  $y_n \rightarrow 0$ , as  $n \rightarrow \infty$ .

**Proof.** If  $R = \frac{c\gamma}{(aca\gamma+ab\gamma+\gamma+c\alpha\beta+c+b\beta)^2} = \frac{1}{4}$ , then we get  $\lambda_1 = \lambda_2 = \frac{1}{2}$ . Let

$$c_{n_1}^{(r)} = \left( \frac{D}{E_1} \frac{Z_1 + (Z_1 - D)(kn_1 + r + 1)}{2Z_1 + (2Z_1 - 2D)(kn_1 + r)} - \frac{F_1}{E_1} \right) \times \left( \frac{D}{E_2} \frac{Z_2 + (Z_2 - D)(kn_1 + r + \lfloor \frac{i-k-1}{2} \rfloor + 1)}{2Z_2 + (2Z_2 - 2D)(kn_1 + r + \lfloor \frac{i-k-1}{2} \rfloor)} - \frac{F_2}{E_2} \right) \quad (4.11)$$

and

$$d_{n_1}^{(r)} = \left( \frac{D}{E_2} \frac{Z_2 + (Z_2 - D)(kn_1 + r + 1)}{2Z_2 + (2Z_2 - 2D)(kn_1 + r)} - \frac{F_2}{E_2} \right) \times \left( \frac{D}{E_1} \frac{Z_1 + (Z_1 - D)(kn_1 + r + \lfloor \frac{i-k-1}{2} \rfloor + 1)}{2Z_1 + (2Z_1 - 2D)(kn_1 + r + \lfloor \frac{i-k-1}{2} \rfloor)} - \frac{F_1}{E_1} \right) \quad (4.12)$$

for every  $n_1 \in \mathbb{N}_0$ ,  $i \in \{1, 2\}$  and  $r \in \mathbb{S}_{k,i}$ . If at least one of the coefficients of  $n_1$  is different from 0, then we have

$$\lim_{n_1 \rightarrow \infty} c_{n_1}^{(r)} = \frac{(D-2F_1)(D-2F_2)}{4E_1E_2} = \lim_{n_1 \rightarrow \infty} d_{n_1}^{(r)} \quad (4.13)$$

for each  $i \in \{1, 2\}$ . Otherwise, when  $\frac{x_{i-2}}{y_{i-k-2}} = \frac{D-2F_1}{2E_1}$  and  $\frac{y_{i-2}}{x_{i-k-2}} = \frac{D-2F_2}{2E_2}$  for  $i \in \{1, 2\}$ , we get the equality (4.13). From (4.13), the results follow from the assumptions in (a) and (b). Now we consider the other cases. For each  $i \in \{1, 2\}$  and sufficiently large  $n_1$ , we obtain

$$\begin{aligned} c_{n_1}^{(r)} &= d_{n_1}^{(r)} = \left( -\frac{F_1}{E_1} + \frac{D}{E_1} \left( \frac{1}{2} + \frac{1}{2kn_1} + \mathcal{O}\left(\frac{1}{n_1^2}\right) \right) \right) \\ &\times \left( -\frac{F_2}{E_2} + \frac{D}{E_2} \left( \frac{1}{2} + \frac{1}{2kn_1} + \mathcal{O}\left(\frac{1}{n_1^2}\right) \right) \right) \\ &= \left( \frac{D-2F_1}{2E_1} + \frac{D}{2E_1kn_1} + \mathcal{O}\left(\frac{1}{n_1^2}\right) \right) \left( \frac{D-2F_2}{2E_2} + \frac{D}{2E_2kn_1} + \mathcal{O}\left(\frac{1}{n_1^2}\right) \right) \\ &= \frac{(D-2F_1)(D-2F_2)}{4E_1E_2} \left( 1 + \frac{\frac{2D(D-F_1-F_2)}{(D-2F_1)(D-2F_2)}}{kn_1} + \mathcal{O}\left(\frac{1}{n_1^2}\right) \right) \\ &= \pm \left( 1 + \frac{1}{\frac{(D-2F_1)(D-2F_2)}{2D(D-F_1-F_2)}kn_1} + \mathcal{O}\left(\frac{1}{n_1^2}\right) \right) \\ &= \pm \exp \left( \frac{1}{\frac{(D-2F_1)(D-2F_2)}{2D(D-F_1-F_2)}kn_1} + \mathcal{O}\left(\frac{1}{n_1^2}\right) \right). \end{aligned} \quad (4.14)$$

From (4.14), by using the fact that  $\sum_{j_1=1}^{n_1} (1/j_1) \rightarrow \infty$  as  $n_1 \rightarrow \infty$ , then the statements easily follow.  $\square$

## 5. Some other applications

Now, we will give theoretical explanations for the formulas of solutions of difference equations systems given in [41] as some applications of the main results in Section 2. First, we consider  $k = l = 1$ ,  $a = \alpha = 0$ . In this case the system (1.5) becomes

$$x_n = \frac{dy_{n-1}x_{n-2}}{bx_{n-2} + cy_{n-1}}, \quad y_n = \frac{\delta x_{n-1}y_{n-2}}{\beta y_{n-2} + \gamma x_{n-1}}, \quad n \in \mathbb{N}_0. \quad (5.1)$$

**5.1. Case  $b = c = d = \beta = \gamma = \delta = 1$**

We will derive the solution forms of the system (5.1) with  $b = c = d = \beta = \gamma = \delta = 1$ , that is, the system

$$x_n = \frac{y_{n-1}x_{n-2}}{x_{n-2} + y_{n-1}}, \quad y_n = \frac{x_{n-1}y_{n-2}}{y_{n-2} + x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{5.2}$$

given in [41], through analytical approach. Also, the general solutions of the system (5.2) are expressed in terms of Fibonacci numbers. Now, to begin with, taking  $b = c = d = \beta = \gamma = \delta = 1$  in (2.12)-(2.13), we have

$$q_{m+1} - q_m + \frac{1}{9}q_{m-1} = 0, \quad m \in \mathbb{N}_0. \tag{5.3}$$

The characteristic equation of (5.3) can be clearly obtained as of the form  $\lambda^2 - \lambda + \frac{1}{9} = 0$ , with the roots  $\lambda_1 = \frac{3+\sqrt{5}}{6} = \frac{1}{3} \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{\hat{\alpha}^2}{3}$  and  $\lambda_2 = \frac{3-\sqrt{5}}{6} = \frac{1}{3} \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{\hat{\beta}^2}{3}$ . On the other hand, taking into account  $A = 3, B_1 = B_2 = 1, C_1 = C_2 = 2$  and the Binet Formula for Fibonacci numbers, we can rewrite the equations in (2.18)-(2.19) as

$$u_{2m+i} = 3 \frac{(\lambda_1 u_{i-2} + 2\lambda_1 - \frac{1}{3})\lambda_1^{m+1} - (\lambda_2 u_{i-2} + 2\lambda_2 - \frac{1}{3})\lambda_2^{m+1}}{(\lambda_1 u_{i-2} + 2\lambda_1 - \frac{1}{3})\lambda_1^m - (\lambda_2 u_{i-2} + 2\lambda_2 - \frac{1}{3})\lambda_2^m} - 2, \tag{5.4}$$

$$v_{2m+i} = 3 \frac{(\lambda_1 v_{i-2} + 2\lambda_1 - \frac{1}{3})\lambda_1^{m+1} - (\lambda_2 v_{i-2} + 2\lambda_2 - \frac{1}{3})\lambda_2^{m+1}}{(\lambda_1 v_{i-2} + 2\lambda_1 - \frac{1}{3})\lambda_1^m - (\lambda_2 v_{i-2} + 2\lambda_2 - \frac{1}{3})\lambda_2^m} - 2, \tag{5.5}$$

for  $m \in \mathbb{N}_0, i \in \{1, 2\}$ .

Using the relations  $\hat{\alpha}\hat{\beta} = -1, \hat{\alpha}^2 + \hat{\beta} = 2, \hat{\beta}^2 + \hat{\alpha} = 2$  in (5.4)-(5.5) we get

$$\begin{aligned} u_{2m+i} &= \frac{(\hat{\alpha}^{2m+4} - \hat{\beta}^{2m+4})u_{i-2} + (\hat{\alpha}^{2m+6} - \hat{\beta}^{2m+6}) - (\hat{\alpha}^{2m+3} - \hat{\beta}^{2m+3}) - (\hat{\alpha}^{2m+2} - \hat{\beta}^{2m+2})}{(\hat{\alpha}^{2m+2} - \hat{\beta}^{2m+2})u_{i-2} + (\hat{\alpha}^{2m+4} - \hat{\beta}^{2m+4}) - (\hat{\alpha}^{2m+1} - \hat{\beta}^{2m+1}) - (\hat{\alpha}^{2m} - \hat{\beta}^{2m})} - 2 \\ &= \frac{F_{2m+4}u_{i-2} + F_{2m+6} - F_{2m+3} - F_{2m+2}}{F_{2m+2}u_{i-2} + F_{2m+4} - F_{2m+1} - F_{2m}} - 2 \\ &= \frac{F_{2m+1}u_{i-2} + F_{2m+2}}{F_{2m+2}u_{i-2} + F_{2m+3}}, \end{aligned} \tag{5.6}$$

$$\begin{aligned} v_{2m+i} &= \frac{(\hat{\alpha}^{2m+4} - \hat{\beta}^{2m+4})v_{i-2} + (\hat{\alpha}^{2m+6} - \hat{\beta}^{2m+6}) - (\hat{\alpha}^{2m+3} - \hat{\beta}^{2m+3}) - (\hat{\alpha}^{2m+2} - \hat{\beta}^{2m+2})}{(\hat{\alpha}^{2m+2} - \hat{\beta}^{2m+2})v_{i-2} + (\hat{\alpha}^{2m+4} - \hat{\beta}^{2m+4}) - (\hat{\alpha}^{2m+1} - \hat{\beta}^{2m+1}) - (\hat{\alpha}^{2m} - \hat{\beta}^{2m})} - 2 \\ &= \frac{F_{2m+4}v_{i-2} + F_{2m+6} - F_{2m+3} - F_{2m+2}}{F_{2m+2}v_{i-2} + F_{2m+4} - F_{2m+1} - F_{2m}} - 2 \\ &= \frac{F_{2m+1}v_{i-2} + F_{2m+2}}{F_{2m+2}v_{i-2} + F_{2m+3}}, \end{aligned} \tag{5.7}$$

where  $F_n$  is  $n$ -th Fibonacci number,  $u_{i-2} = \frac{x_{i-2}}{y_{i-3}}, v_{i-2} = \frac{y_{i-2}}{x_{i-3}}$ . From (2.1), (5.2), and (5.6), we get that, for  $m \in \mathbb{N}_0$  and  $i \in \{1, 2\}$ ,

$$\begin{aligned} u_{2m+1} &= \frac{F_{2m+1}u_{-1} + F_{2m+2}}{F_{2m+2}u_{-1} + F_{2m+3}} \\ &= \frac{x_{-1}F_{2m+1} + y_{-2}F_{2m+2}}{x_{-1}F_{2m+2} + y_{-2}F_{2m+3}}, \end{aligned} \tag{5.8}$$

$$\begin{aligned} u_{2m+2} &= \frac{F_{2m+1}u_0 + F_{2m+2}}{F_{2m+2}u_0 + F_{2m+3}} \\ &= \frac{y_{-1}F_{2m+2} + x_{-2}F_{2m+3}}{y_{-1}F_{2m+3} + x_{-2}F_{2m+4}}. \end{aligned} \tag{5.9}$$

Similarly, from (2.1), (5.2), and (5.7), we get that, for  $m \in \mathbb{N}_0$  and  $i \in \{1, 2\}$ ,

$$\begin{aligned} v_{2m+1} &= \frac{F_{2m+1}v_{-1} + F_{2m+2}}{F_{2m+2}v_{-1} + F_{2m+3}} \\ &= \frac{y_{-1}F_{2m+1} + x_{-2}F_{2m+2}}{y_{-1}F_{2m+2} + x_{-2}F_{2m+3}}, \end{aligned} \quad (5.10)$$

$$\begin{aligned} v_{2m+2} &= \frac{F_{2m+1}v_0 + F_{2m+2}}{F_{2m+2}v_0 + F_{2m+3}} \\ &= \frac{x_{-1}F_{2m+2} + y_{-2}F_{2m+3}}{x_{-1}F_{2m+3} + y_{-2}F_{2m+4}}. \end{aligned} \quad (5.11)$$

By substituting the formulas in (5.8)-(5.11) into (2.27)-(2.28) and changing indexes, we have the following results.

**Theorem 5.1.** *Assume that  $(x_n, y_n)_{n \geq -2}$  is a well-defined solution of the system (5.2). Then the following results are true.*

$$x_{2m+1} = \frac{x_{-1}y_{-2}}{y_{-2}F_{2m+3} + x_{-1}F_{2m+2}}, \quad x_{2m+2} = \frac{y_{-1}x_{-2}}{y_{-1}F_{2m+3} + x_{-2}F_{2m+4}}, \quad m \geq -1,$$

$$y_{2m+1} = \frac{y_{-1}x_{-2}}{x_{-2}F_{2m+3} + y_{-1}F_{2m+2}}, \quad y_{2m+2} = \frac{x_{-1}y_{-2}}{x_{-1}F_{2m+3} + y_{-2}F_{2m+4}}, \quad m \geq -1,$$

where  $\{F_n\}_{n=0}^\infty = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$ ,  $F_{-1} = 1$ .

## 5.2. Case $b = -c = d = -\beta = -\gamma = \delta = 1$

We will derive the solution forms of the system (5.1) with  $b = -c = d = -\beta = -\gamma = \delta = 1$ , that is, the system

$$x_n = \frac{y_{n-1}x_{n-2}}{x_{n-2} - y_{n-1}}, \quad y_n = \frac{x_{n-1}y_{n-2}}{-y_{n-2} - x_{n-1}}, \quad n \in \mathbb{N}_0, \quad (5.12)$$

given in [41], through analytical approach. Also, the general solutions of the system (5.12) are expressed in terms of Fibonacci numbers. Now, to begin with, taking  $b = -c = d = -\beta = -\gamma = \delta = 1$  in (2.12)-(2.13), we have

$$q_{m+1} - q_m + \frac{1}{9}q_{m-1} = 0, \quad m \in \mathbb{N}_0. \quad (5.13)$$

The characteristic equation of (5.13) can be clearly obtained as of the form  $\lambda^2 - \lambda + \frac{1}{9} = 0$  with the roots  $\lambda_1 = \frac{3+\sqrt{5}}{6} = \frac{1}{3} \left( \frac{1+\sqrt{5}}{2} \right)^2 = \frac{\hat{\alpha}^2}{3}$  and  $\lambda_2 = \frac{3-\sqrt{5}}{6} = \frac{1}{3} \left( \frac{1-\sqrt{5}}{2} \right)^2 = \frac{\hat{\beta}^2}{3}$ . On the other hand, taking into account  $A = -3, B_1 = -1, B_2 = 1, C_1 = C_2 = -2$  and the Binet Formula for Fibonacci numbers, then we can rewrite the equations in (2.18)-(2.19) as

$$u_{2m+i} = 3 \frac{(-\lambda_1 u_{i-2} - 2\lambda_1 + \frac{1}{3})\lambda_1^{m+1} - (-\lambda_2 u_{i-2} - 2\lambda_2 + \frac{1}{3})\lambda_2^{m+1}}{(-\lambda_1 u_{i-2} - 2\lambda_1 + \frac{1}{3})\lambda_1^m - (-\lambda_2 u_{i-2} - 2\lambda_2 + \frac{1}{3})\lambda_2^m} - 2, \quad (5.14)$$

$$v_{2m+i} = -3 \frac{(\lambda_1 v_{i-2} - 2\lambda_1 + \frac{1}{3})\lambda_1^{m+1} - (\lambda_2 v_{i-2} - 2\lambda_2 + \frac{1}{3})\lambda_2^{m+1}}{(\lambda_1 v_{i-2} - 2\lambda_1 + \frac{1}{3})\lambda_1^m - (\lambda_2 v_{i-2} - 2\lambda_2 + \frac{1}{3})\lambda_2^m} + 2, \quad (5.15)$$

for  $m \in \mathbb{N}_0, i \in \{1, 2\}$ .

Using the relations  $\hat{\alpha}\hat{\beta} = -1, \hat{\alpha}^2 + \hat{\beta} = 2, \hat{\beta}^2 + \hat{\alpha} = 2$  in (5.14)-(5.15) we get

$$\begin{aligned} u_{2m+i} &= \frac{-(\hat{\alpha}^{2m+4} - \hat{\beta}^{2m+4})u_{i-2} - (\hat{\alpha}^{2m+6} - \hat{\beta}^{2m+6}) + (\hat{\alpha}^{2m+3} - \hat{\beta}^{2m+3}) + (\hat{\alpha}^{2m+2} - \hat{\beta}^{2m+2})}{-(\hat{\alpha}^{2m+2} - \hat{\beta}^{2m+2})u_{i-2} - (\hat{\alpha}^{2m+4} - \hat{\beta}^{2m+4}) + (\hat{\alpha}^{2m+1} - \hat{\beta}^{2m+1}) + (\hat{\alpha}^{2m} - \hat{\beta}^{2m})} - 2 \\ &= \frac{-F_{2m+4}u_{i-2} - F_{2m+6} + F_{2m+3} + F_{2m+2}}{-F_{2m+2}u_{i-2} - F_{2m+4} + F_{2m+1} + F_{2m}} - 2 \\ &= \frac{-F_{2m+1}u_{i-2} - F_{2m+2}}{-F_{2m+2}u_{i-2} - F_{2m+3}}, \end{aligned} \quad (5.16)$$

$$\begin{aligned}
 v_{2m+i} &= -\frac{(\widehat{\alpha}^{2m+4} - \widehat{\beta}^{2m+4})v_{i-2} - (\widehat{\alpha}^{2m+6} - \widehat{\beta}^{2m+6}) + (\widehat{\alpha}^{2m+3} - \widehat{\beta}^{2m+3}) + (\widehat{\alpha}^{2m+2} - \widehat{\beta}^{2m+2})}{(\widehat{\alpha}^{2m+2} - \widehat{\beta}^{2m+2})v_{i-2} - (\widehat{\alpha}^{2m+4} - \widehat{\beta}^{2m+4}) + (\widehat{\alpha}^{2m+1} - \widehat{\beta}^{2m+1}) + (\widehat{\alpha}^{2m} - \widehat{\beta}^{2m})} + 2 \\
 &= -\frac{F_{2m+4}v_{i-2} - F_{2m+6} + F_{2m+3} + F_{2m+2}}{F_{2m+2}v_{i-2} - F_{2m+4} + F_{2m+1} + F_{2m}} + 2 \\
 &= \frac{-F_{2m+1}v_{i-2} + F_{2m+2}}{F_{2m+2}v_{i-2} - F_{2m+3}},
 \end{aligned}
 \tag{5.17}$$

where  $F_n$  is  $n$ -th Fibonacci number,  $u_{i-2} = \frac{x_{i-2}}{y_{i-3}}$ ,  $v_{i-2} = \frac{y_{i-2}}{x_{i-3}}$ . From (2.1), (5.12), and (5.16), we get that, for  $m \in \mathbb{N}_0$  and  $i \in \{1, 2\}$ ,

$$\begin{aligned}
 u_{2m+1} &= \frac{-F_{2m+1}u_{-1} - F_{2m+2}}{-F_{2m+2}u_{-1} - F_{2m+3}} \\
 &= \frac{-x_{-1}F_{2m+1} - y_{-2}F_{2m+2}}{-x_{-1}F_{2m+2} - y_{-2}F_{2m+3}},
 \end{aligned}
 \tag{5.18}$$

$$\begin{aligned}
 u_{2m+2} &= \frac{-F_{2m+1}u_0 - F_{2m+2}}{-F_{2m+2}u_0 - F_{2m+3}} \\
 &= \frac{y_{-1}F_{2m+2} - x_{-2}F_{2m+3}}{y_{-1}F_{2m+3} - x_{-2}F_{2m+4}}.
 \end{aligned}
 \tag{5.19}$$

Similarly, from (2.1), (5.12), and (5.17), we get that, for  $m \in \mathbb{N}_0$  and  $i \in \{1, 2\}$ ,

$$\begin{aligned}
 v_{2m+1} &= \frac{-F_{2m+1}v_{-1} + F_{2m+2}}{F_{2m+2}v_{-1} - F_{2m+3}} \\
 &= \frac{-y_{-1}F_{2m+1} + x_{-2}F_{2m+2}}{y_{-1}F_{2m+2} - x_{-2}F_{2m+3}},
 \end{aligned}
 \tag{5.20}$$

$$\begin{aligned}
 v_{2m+2} &= \frac{-F_{2m+1}v_0 + F_{2m+2}}{F_{2m+2}v_0 - F_{2m+3}} \\
 &= \frac{-x_{-1}F_{2m+2} - y_{-2}F_{2m+3}}{x_{-1}F_{2m+3} + y_{-2}F_{2m+4}}.
 \end{aligned}
 \tag{5.21}$$

By substituting the formulas in (5.18)-(5.21) into (2.27)-(2.28) and changing indexes, we have the following results.

**Theorem 5.2.** Assume that  $(x_n, y_n)_{n \geq -2}$  is a well-defined solution of the system (5.12). Then the following results are true.

$$x_{2m+1} = \frac{(-1)^{m+1}x_{-1}y_{-2}}{y_{-2}F_{2m+3} + x_{-1}F_{2m+2}}, \quad x_{2m+2} = \frac{(-1)^{m+1}y_{-1}x_{-2}}{-y_{-1}F_{2m+3} + x_{-2}F_{2m+4}}, \quad m \geq -1,$$

$$y_{2m+1} = \frac{(-1)^{m+1}y_{-1}x_{-2}}{x_{-2}F_{2m+3} - y_{-1}F_{2m+2}}, \quad y_{2m+2} = \frac{(-1)^m x_{-1}y_{-2}}{x_{-1}F_{2m+3} + y_{-2}F_{2m+4}}, \quad m \geq -1,$$

where  $\{F_n\}_{n=0}^\infty = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$ ,  $F_{-1} = 1$ .

**5.3. Case  $b = -c = d = \beta = \gamma = \delta = 1$**

We will derive the solution forms of the system (5.1) with  $b = -c = d = \beta = \gamma = \delta = 1$ , that is, the system

$$x_n = \frac{y_{n-1}x_{n-2}}{x_{n-2} - y_{n-1}}, \quad y_n = \frac{x_{n-1}y_{n-2}}{y_{n-2} + x_{n-1}}, \quad n \in \mathbb{N}_0,
 \tag{5.22}$$

given in [41], through analytical approach. Also, the general solutions of the system (5.22) are expressed in terms of Fibonacci numbers. Now, to begin with, taking  $b = -c = d = \beta = \gamma = \delta = 1$  in (2.12)-(2.13), we have

$$q_{m+1} - q_m - q_{m-1} = 0, \quad m \in \mathbb{N}_0.
 \tag{5.23}$$

The characteristic equation of (5.23) can be clearly obtained as of the form  $\lambda^2 - \lambda - 1 = 0$  with the roots  $\lambda_1 = \frac{1+\sqrt{5}}{2} = \hat{\alpha}$  and  $\lambda_2 = \frac{1-\sqrt{5}}{2} = \hat{\beta}$ . On the other hand, taking into account  $A = 1, B_1 = 1, B_2 = -1, C_1 = 0, C_2 = 2$  and the Binet Formula for Fibonacci numbers, we can rewrite the equations in (2.18)-(2.19) as for,  $m \in \mathbb{N}_0$  and  $i \in \{1, 2\}$ ,

$$u_{2m+i} = \frac{(\lambda_1 u_{i-2} + 1)\lambda_1^{m+1} - (\lambda_2 u_{i-2} + 1)\lambda_2^{m+1}}{(\lambda_1 u_{i-2} + 1)\lambda_1^m - (\lambda_2 u_{i-2} + 1)\lambda_2^m}, \quad (5.24)$$

$$v_{2m+i} = -\frac{(-\lambda_1 v_{i-2} + 2\lambda_1 + 1)\lambda_1^{m+1} - (-\lambda_2 v_{i-2} + 2\lambda_2 + 1)\lambda_2^{m+1}}{(-\lambda_1 v_{i-2} + 2\lambda_1 + 1)\lambda_1^m - (-\lambda_2 v_{i-2} + 2\lambda_2 + 1)\lambda_2^m} + 2, \quad (5.25)$$

for  $m \in \mathbb{N}_0, i \in \{1, 2\}$ .

Using the relations  $\hat{\alpha}\hat{\beta} = -1, \hat{\alpha}^2 + \hat{\beta} = 2, \hat{\beta}^2 + \hat{\alpha} = 2$  in (5.24)-(5.25) we get

$$\begin{aligned} u_{2m+i} &= \frac{(\hat{\alpha}^{m+2} - \hat{\beta}^{m+2})u_{i-2} + (\hat{\alpha}^{m+1} - \hat{\beta}^{m+1})}{(\hat{\alpha}^{m+1} - \hat{\beta}^{m+1})u_{i-2} + (\hat{\alpha}^m - \hat{\beta}^m)} \\ &= \frac{F_{m+2}u_{i-2} + F_{m+1}}{F_{m+1}u_{i-2} + F_m}, \end{aligned} \quad (5.26)$$

$$\begin{aligned} v_{2m+i} &= \frac{(\hat{\alpha}^{m+2} - \hat{\beta}^{m+2})v_{i-2} - (\hat{\alpha}^{m+4} - \hat{\beta}^{m+4})}{-(\hat{\alpha}^{m+1} - \hat{\beta}^{m+1})v_{i-2} + (\hat{\alpha}^{m+3} - \hat{\beta}^{m+3})} + 2 \\ &= \frac{F_{m+2}v_{i-2} - F_{m+4}}{-F_{m+1}v_{i-2} + F_{m+3}} + 2 \\ &= \frac{-F_{m-1}v_{i-2} + F_{m+1}}{-F_{m+1}v_{i-2} + F_{m+3}}, \end{aligned} \quad (5.27)$$

where  $F_n$  is  $n$ -th Fibonacci number,  $u_{i-2} = \frac{x_{i-2}}{y_{i-3}}$ ,  $v_{i-2} = \frac{y_{i-2}}{x_{i-3}}$ . From (2.1), (5.22) and (5.26), we get that, for  $m \in \mathbb{N}_0$  and  $i \in \{1, 2\}$ ,

$$\begin{aligned} u_{2m+1} &= \frac{F_{m+2}u_{-1} + F_{m+1}}{F_{m+1}u_{-1} + F_m} \\ &= \frac{x_{-1}F_{m+2} + y_{-2}F_{m+1}}{x_{-1}F_{m+1} + y_{-2}F_m}, \end{aligned} \quad (5.28)$$

$$\begin{aligned} u_{2m+2} &= \frac{F_{m+2}u_0 + F_{m+1}}{F_{m+1}u_0 + F_m} \\ &= \frac{-y_{-1}F_{m+1} + x_{-2}F_{m+3}}{-y_{-1}F_m + x_{-2}F_{m+2}}. \end{aligned} \quad (5.29)$$

Similarly, from (2.1), (5.22), and (5.27), we get that, for  $m \in \mathbb{N}_0$  and  $i \in \{1, 2\}$ ,

$$\begin{aligned} v_{2m+1} &= \frac{-F_{m-1}v_{-1} + F_{m+1}}{-F_{m+1}v_{-1} + F_{m+3}} \\ &= \frac{-y_{-1}F_{m-1} + x_{-2}F_{m+1}}{-y_{-1}F_{m+1} + x_{-2}F_{m+3}}, \end{aligned} \quad (5.30)$$

$$\begin{aligned} v_{2m+2} &= \frac{-F_{m-1}v_0 + F_{m+1}}{-F_{m+1}v_0 + F_{m+3}} \\ &= \frac{x_{-1}F_{m+1} + y_{-2}F_m}{x_{-1}F_{m+3} + y_{-2}F_{m+2}}. \end{aligned} \quad (5.31)$$

By substituting the formulas in (5.28)-(5.31) into (2.27)-(2.28) and changing indexes, we have the following results.



**Theorem 5.3.** Assume that  $(x_n, y_n)_{n \geq -2}$  is a well-defined solution of the system (5.22). Then the following results are true.

$$x_{2m+1} = \frac{x_{-1}y_{-2}}{y_{-2}F_m + x_{-1}F_{m+1}}, \quad x_{2m+2} = \frac{y_{-1}x_{-2}}{-y_{-1}F_m + x_{-2}F_{m+2}}, \quad m \geq -1,$$

$$y_{2m+1} = \frac{y_{-1}x_{-2}}{x_{-2}F_{m+3} - y_{-1}F_{m+1}}, \quad y_{2m+2} = \frac{x_{-1}y_{-2}}{x_{-1}F_{m+3} + y_{-2}F_{m+2}}, \quad m \geq -1,$$

where  $\{F_n\}_{n=0}^\infty = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$ ,  $F_{-1} = 1$ .

**5.4. Case  $b = c = d = \beta = -\gamma = \delta = 1$**

We will derive the solution forms of the system (5.1) with  $b = c = d = \beta = -\gamma = \delta = 1$ , that is, the system

$$x_n = \frac{y_{n-1}x_{n-2}}{x_{n-2} + y_{n-1}}, \quad y_n = \frac{x_{n-1}y_{n-2}}{y_{n-2} - x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{5.32}$$

given in [41], through analytical approach. Also, the general solutions of the system (5.32) are expressed in terms of Fibonacci numbers. Now, to begin with, taking  $b = c = d = \beta = -\gamma = \delta = 1$  in (2.12)-(2.13), we have

$$q_{m+1} - q_m - q_{m-1} = 0, \quad m \in \mathbb{N}_0. \tag{5.33}$$

The characteristic equation of (5.33) can be clearly obtained as of the form  $\lambda^2 - \lambda - 1 = 0$  with the roots  $\lambda_1 = \frac{1+\sqrt{5}}{2} = \hat{\alpha}$  and  $\lambda_2 = \frac{1-\sqrt{5}}{2} = \hat{\beta}$ . On the other hand, taking into account  $A = 1, B_1 = -1, B_2 = 1, C_1 = 2, C_2 = 0$  and the Binet Formula for Fibonacci numbers, then we can rewrite the equations in (2.18)-(2.19) as

$$u_{2m+i} = -\frac{(-\lambda_1 u_{i-2} + 2\lambda_1 + 1)\lambda_1^{m+1} - (-\lambda_2 u_{i-2} + 2\lambda_2 + 1)\lambda_2^{m+1}}{(-\lambda_1 u_{i-2} + 2\lambda_1 + 1)\lambda_1^m - (-\lambda_2 u_{i-2} + 2\lambda_2 + 1)\lambda_2^m} + 2, \tag{5.34}$$

$$v_{2m+i} = \frac{(\lambda_1 v_{i-2} + 1)\lambda_1^{m+1} - (\lambda_2 v_{i-2} + 1)\lambda_2^{m+1}}{(\lambda_1 v_{i-2} + 1)\lambda_1^m - (\lambda_2 v_{i-2} + 1)\lambda_2^m}, \tag{5.35}$$

for  $m \in \mathbb{N}_0, i \in \{1, 2\}$ .

Using the relations  $\hat{\alpha}\hat{\beta} = -1, \hat{\alpha}^2 + \hat{\beta} = 2, \hat{\beta}^2 + \hat{\alpha} = 2$  in (5.14)-(5.15) we get

$$\begin{aligned} u_{2m+i} &= \frac{(\hat{\alpha}^{m+2} - \hat{\beta}^{m+2})u_{i-2} - (\hat{\alpha}^{m+4} - \hat{\beta}^{m+4})}{-(\hat{\alpha}^{m+1} - \hat{\beta}^{m+1})u_{i-2} + (\hat{\alpha}^{m+3} - \hat{\beta}^{m+3})} + 2 \\ &= \frac{F_{m+2}u_{i-2} - F_{m+4}}{-F_{m+1}u_{i-2} + F_{m+3}} + 2 \\ &= \frac{-F_{m-1}u_{i-2} + F_{m+1}}{-F_{m+1}u_{i-2} + F_{m+3}}, \end{aligned} \tag{5.36}$$

$$\begin{aligned} v_{2m+i} &= \frac{(\hat{\alpha}^{m+2} - \hat{\beta}^{m+2})v_{i-2} + (\hat{\alpha}^{m+1} - \hat{\beta}^{m+1})}{(\hat{\alpha}^{m+1} - \hat{\beta}^{m+1})v_{i-2} + (\hat{\alpha}^m - \hat{\beta}^m)} \\ &= \frac{F_{m+2}v_{i-2} + F_{m+1}}{F_{m+1}v_{i-2} + F_m}, \end{aligned} \tag{5.37}$$

where  $F_n$  is  $n$ -th Fibonacci number,  $u_{i-2} = \frac{x_{i-2}}{y_{i-3}}, v_{i-2} = \frac{y_{i-2}}{x_{i-3}}$ . From (2.1), (5.32), and (5.36), we get that, for  $m \in \mathbb{N}_0$  and  $i \in \{1, 2\}$ ,

$$\begin{aligned} u_{2m+1} &= \frac{-F_{m-1}u_{-1} + F_{m+1}}{-F_{m+1}u_{-1} + F_{m+3}} \\ &= \frac{-x_{-1}F_{m-1} + y_{-2}F_{m+1}}{-x_{-1}F_{m+1} + y_{-2}F_{m+3}}, \end{aligned} \tag{5.38}$$

$$\begin{aligned}
 u_{2m+2} &= \frac{-F_{m-1}u_0 + F_{m+1}}{-F_{m+1}u_0 + F_{m+3}} \\
 &= \frac{y_{-1}F_{m+1} + x_{-2}F_m}{y_{-1}F_{m+3} + x_{-2}F_{m+2}}.
 \end{aligned} \tag{5.39}$$

Similarly, from (2.1), (5.32), and (5.37), we get that, for  $m \in \mathbb{N}_0$  and  $i \in \{1, 2\}$ ,

$$\begin{aligned}
 v_{2m+1} &= \frac{F_{m+2}v_{-1} + F_{m+1}}{F_{m+1}v_{-1} + F_m} \\
 &= \frac{y_{-1}F_{m+2} + x_{-2}F_{m+1}}{y_{-1}F_{m+1} + x_{-2}F_m},
 \end{aligned} \tag{5.40}$$

$$\begin{aligned}
 v_{2m+2} &= \frac{F_{m+2}v_0 + F_{m+1}}{F_{m+1}v_0 + F_m} \\
 &= \frac{-x_{-1}F_{m+1} + y_{-2}F_{m+3}}{-x_{-1}F_m + y_{-2}F_{m+2}}.
 \end{aligned} \tag{5.41}$$

By substituting the formulas in (5.38)-(5.41) into (2.27)-(2.28) and changing indexes, we have the following results.

**Theorem 5.4.** Assume that  $(x_n, y_n)_{n \geq -2}$  is a well-defined solution of the system (5.32). Then the following results are true.

$$x_{2m+1} = \frac{x_{-1}y_{-2}}{y_{-2}F_{m+3} - x_{-1}F_{m+1}}, \quad x_{2m+2} = \frac{y_{-1}x_{-2}}{y_{-1}F_{m+3} + x_{-2}F_{m+2}}, \quad m \geq -1,$$

$$y_{2m+1} = \frac{y_{-1}x_{-2}}{x_{-2}F_m + y_{-1}F_{m+1}}, \quad y_{2m+2} = \frac{x_{-1}y_{-2}}{-x_{-1}F_m + y_{-2}F_{m+2}}, \quad m \geq -1,$$

where  $\{F_n\}_{n=0}^\infty = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$ ,  $F_{-1} = 1$ .

**Acknowledgment.** The authors are thankful to the editor and reviewers for their constructive review.

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