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ON A THREE-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS†

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Abstract. Consider the three-dimensional system of difference equations

$$x_{n+1}=\frac{\prod_{j=0}^{k} z_{n-3j}}{\prod_{j=1}^{k} x_{n-(3j-1)}\left(a_n+b_n\prod_{j=0}^{k} z_{n-3j}\right)},$$

$$y_{n+1}=\frac{\prod_{j=0}^{k} x_{n-3j}}{\prod_{j=1}^{k} y_{n-(3j-1)}\left(c_n+d_n\prod_{j=0}^{k} x_{n-3j}\right)},$$

$$z_{n+1}=\frac{\prod_{j=0}^{k} y_{n-3j}}{\prod_{j=1}^{k} z_{n-(3j-1)}\left(e_n+f_n\prod_{j=0}^{k} y_{n-3j}\right)},\ n\in\mathbb{N}_0,$$

where $k\in\mathbb{N}_0$, the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(e_n)_{n\in\mathbb{N}_0}$, $(f_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-3k}, x_{-3k+1}, $\dots$, x_0, y_{-3k}, y_{-3k+1}, $\dots$, y_0, z_{-3k}, z_{-3k+1}, $\dots$, z_0 are real numbers.

In this work, we give explicit formulas for the well defined solutions of the above system. Also, the forbidden set of solution of the system is found. For the constant case, a result on the existence of periodic solutions is provided and the asymptotic behavior of the solutions is investigated in detail.

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1. Introduction

First, remind that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, stand for natural, non-negative integer, integer, real and complex numbers, respectively. If $m,n\in\mathbb{Z},\ m\le n$ the notation $i=\overline{m,n}$ stands for $\{i\in\mathbb{Z}: m\le i\le n\}$.

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In this paper, we consider the following three-dimensional system,

$$
\begin{aligned}
x_{n+1} &= \frac{\prod_{j=0}^{k} z_{n-3j}}{\prod_{j=1}^{k} x_{n-(3j-1)}\left(a_n + b_n \prod_{j=0}^{k} z_{n-3j}\right)}, \\
y_{n+1} &= \frac{\prod_{j=0}^{k} x_{n-3j}}{\prod_{j=1}^{k} y_{n-(3j-1)}\left(c_n + d_n \prod_{j=0}^{k} x_{n-3j}\right)}, \\
z_{n+1} &= \frac{\prod_{j=0}^{k} y_{n-3j}}{\prod_{j=1}^{k} z_{n-(3j-1)}\left(e_n + f_n \prod_{j=0}^{k} y_{n-3j}\right)},\ n \in \mathbb{N}_0,
\end{aligned} \tag{1}
$$

where $k \in \mathbb{N}_0$ the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(e_n)_{n\in\mathbb{N}_0}$, $(f_n)_{n\in\mathbb{N}_0}$ are real and the initial values x_{-3k}, x_{-3k+1}, $\dots$, x_0, y_{-3k}, y_{-3k+1}, $\dots$, y_0, z_{-3k}, z_{-3k+1}, $\dots$, z_0 are real numbers.

Difference equations emerge from the study of the evolution of naturally occurring events. The theory of difference equations systems and difference equations greatly improved until today. Recently, there has been great interest in studying difference equations systems. Because difference equations and their systems are used to describes real discrete models in various branches of modern sciences such as biology, economics, physics, engineering genetics, psychology, control theory. In addition, the applications of difference equations systems are rapidly increasing to aforementioned fields. There is no doubt that the theory of difference equations will proceed to play an important role in mathematics. Especially, non-linear difference equations and their systems play an important role in applications. These difference equations and their systems often arise as mathematical model of a problem. In such a case, solutions of the model is examined by means of mathematical methods. Therefore, the non-linear difference equations are a rich area of study in mathematics. Consequently, studying the solutions of difference equations and its qualitative behaviors have become focus topics for research. The main problem of theory of difference equations is to state behaviour of the solutions of difference equations. There are some methods of doing this. The most basic and classical of these methods is undoubtedly to find a closed formula for the solutions of equations. By doing so, one can acquire more concrete results. Most non-linear difference equations and systems of difference equations cannot be solved. However, by the help of appropriate transformations, some types can be transformed into linear difference equations or their systems which can be generally solved in closed form.

Solving non-linear difference equations and their systems is a very hot topics that continue to attract the attention of a wide range of researchers, we can consult the following papers [1,2,11,13–17,19–22,26,28,29] to see several models of difference equations and systems that are solved in closed form, but also to understand procedures used in solving such equations and systems.
Many authors solved or investigated global behavior of the case $k = 0$ in system

(1), which is a two-dimensional system in [3–5, 27]. Also the global asymptotic behavior of solutions of difference equations or two and three dimensional systems where investigated in several studies, see for example [6, 7, 12, 18, 23–25].

In [10], El-Metwally et al., obtained the solutions of the following fourth order difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}\left(\pm 1 \pm x_n x_{n-3}\right)},\ n \in \mathbb{N}_0, \tag{2}$$

In [9] and as extension of the work in [10], the authors solve the two-dimensional system of difference equations

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}\left(\pm 1 \pm x_n y_{n-3}\right)},\ y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}\left(\pm 1 \pm y_n x_{n-3}\right)}\ n \in \mathbb{N}_0, \tag{3}$$

Clearly equation (2) is a particular case of the one dimensional version of our system (1) for $k = 1$.

In an earlier paper, Haddad et al., in [14], deal with the following system of difference equations

$$\begin{aligned} x_{n+1} &= \frac{\prod_{j=0}^{k} y_{n-2j}}{\prod_{j=1}^{k} x_{n-(2j-1)}\left(a_n + b_n \prod_{j=0}^{k} y_{n-2j}\right)}, \\ y_{n+1} &= \frac{\prod_{j=0}^{k} x_{n-2j}}{\prod_{j=1}^{k} y_{n-(2j-1)}\left(\alpha_n + \beta_n \prod_{j=0}^{k} x_{n-2j}\right)},\ n \in \mathbb{N}_0. \end{aligned} \tag{4}$$

The authors showed that the system (4) is solvable in closed form and presented formulas for the solution.

Motivated by all of these results, we solve the system (1) in explicit form and we describe the forbidden set for the initial values. For the coefficients are constant case we show existence of periodic solutions and we investigate the asymptotic behavior of well-defined solutions.

To solve system (1), we will use a change of variable to transform our system to some first order linear systems. For this purpose we will use the following very well known result, see for example [8].

Lemma 1.1. *Consider the linear difference equation*

$$y_{n+1} = a_n y_n + b_n,\ n \in \mathbb{N}_0.$$

Then,

$$y_n = \left(\prod_{i=0}^{n-1} a_i\right) y_0 + \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} a_i\right) b_r.$$

Moreover if a_n and b_n are constants, that is $a_n = a$ and $b_n = b$, then

$$y_n = \begin{cases} a^n y_0 + \frac{a^n - 1}{a - 1} b,\ n \in \mathbb{N}_0, & a \neq 1, \\ y_0 + bn,\ n \in \mathbb{N}_0, & a = 1, \end{cases}$$

where as usual, $\prod_{j=i}^{m} \alpha_j = 1$ and $\sum_{j=i}^{m} \beta_j = 0$, for all $m < i$.

2. Form of the solutions of system (1)

In the following, we obtain the form of the solutions of the system (1). Firstly, we recall that we mean by a well defined solution of the system (1), a solution which satisfies:

$$\prod_{j=1}^{k} x_{n-(3j-1)} \left(a_n + b_n \prod_{j=0}^{k} z_{n-3j} \right) \neq 0,\ n \in \mathbb{N}_0,$$
$$\prod_{j=1}^{k} y_{n-(3j-1)} \left(c_n + d_n \prod_{j=0}^{k} x_{n-3j} \right) \neq 0,\ n \in \mathbb{N}_0,$$

and

$$\prod_{j=1}^{k} z_{n-(3j-1)} \left(e_n + f_n \prod_{j=0}^{k} y_{n-3j} \right) \neq 0,\ n \in \mathbb{N}_0.$$

Putting

$$u_n = \frac{1}{\prod_{j=0}^{k} x_{n-3j}},\ v_n = \frac{1}{\prod_{j=0}^{k} y_{n-3j}},\ w_n = \frac{1}{\prod_{j=0}^{k} z_{n-3j}},\ n \in \mathbb{N}_0, \tag{5}$$

then system (1) becomes

$$u_{n+1} = a_n w_n + b_n,\ v_{n+1} = c_n u_n + d_n,\ w_{n+1} = e_n v_n + f_n,\ n \in \mathbb{N}_0. \tag{6}$$

From (6) we get

$$u_{n+3} = a_{n+2}e_{n+1}c_n u_n + a_{n+2}e_{n+1}d_n + a_{n+2}f_{n+1} + b_{n+2},\ n \in \mathbb{N}_0, \tag{7}$$
$$v_{n+3} = c_{n+2}a_{n+1}e_n v_n + c_{n+2}a_{n+1}f_n + c_{n+2}b_{n+1} + d_{n+2},\ n \in \mathbb{N}_0, \tag{8}$$
$$w_{n+3} = e_{n+2}c_{n+1}a_n w_n + e_{n+2}c_{n+1}b_n + e_{n+2}d_{n+1} + f_{n+2},\ n \in \mathbb{N}_0. \tag{9}$$

If we apply the decomposition of indices $n \to 3n+j$ for $n \in \mathbb{N}_0$ and $j \in \{0,1,2\}$, to equations in (7)-(9), for $n \in \mathbb{N}_0$ they become

$$\begin{aligned} u_{3(n+1)+j} &= a_{3n+j+2}e_{3n+j+1}c_{3n+j}u_{3n+j} + a_{3n+j+2}e_{3n+j+1}d_{3n+j} \\ &+ a_{3n+j+2}f_{3n+j+1} + b_{3n+j+2}, \end{aligned} \tag{10}$$

$$\begin{aligned} v_{3(n+1)+j} &= c_{3n+j+2}a_{3n+j+1}e_{3n+j}v_{3n+j} + c_{3n+j+2}a_{3n+j+1}f_{3n+j} \\ &+ c_{3n+j+2}b_{3n+j+1} + d_{3n+j+2}, \end{aligned} \tag{11}$$

$$\begin{aligned} w_{3(n+1)+j} &= e_{3n+j+2}c_{3n+j+1}a_{3n+j}w_{3n+j} + e_{3n+j+2}c_{3n+j+1}b_{3n+j} \\ &+ e_{3n+j+2}d_{3n+j+1} + f_{3n+j+2}. \end{aligned} \tag{12}$$

Let $u_n^{(j)} = u_{3n+j}$, $v_n^{(j)} = v_{3n+j}$, $w_n^{(j)} = w_{3n+j}$ for $n \in \mathbb{N}_0$ and $j \in \{0,1,2\}$ and

$$\begin{aligned} A_n^{(j)} &= a_{3n+j+2}e_{3n+j+1}c_{3n+j}, \\ B_n^{(j)} &= a_{3n+j+2}e_{3n+j+1}d_{3n+j} + a_{3n+j+2}f_{3n+j+1} + b_{3n+j+2}, \end{aligned} \tag{13}$$

$$\begin{aligned} C_n^{(j)} &= c_{3n+j+2}a_{3n+j+1}e_{3n+j}, \\ D_n^{(j)} &= c_{3n+j+2}a_{3n+j+1}f_{3n+j}+c_{3n+j+2}b_{3n+j+1}+d_{3n+j+2}, \end{aligned} \tag{14}$$

$$\begin{aligned} E_n^{(j)} &= e_{3n+j+2}c_{3n+j+1}a_{3n+j}, \\ F_n^{(j)} &= e_{3n+j+2}c_{3n+j+1}b_{3n+j}+e_{3n+j+2}d_{3n+j+1}+f_{3n+j+2}. \end{aligned} \tag{15}$$

Then equations in (10)-(12) can be written as the following

$$u_{n+1}^{(j)} = A_n^{(j)}u_n^{(j)} + B_n^{(j)},\ n \in \mathbb{N}_0, \tag{16}$$

$$v_{n+1}^{(j)} = C_n^{(j)}v_n^{(j)} + D_n^{(j)},\ n \in \mathbb{N}_0, \tag{17}$$

$$w_{n+1}^{(j)} = E_n^{(j)}w_n^{(j)} + F_n^{(j)},\ n \in \mathbb{N}_0, \tag{18}$$

for $j \in \{0,1,2\}$.
From (16)-(18) and Lemma 1.1, we have

$$u_n^{(j)} = \left(\prod_{j_1=0}^{n-1} A_{j_1}^{(j)}\right) u_0^{(j)} + \sum_{j_1=0}^{n-1}\left(\prod_{i=j_1+1}^{n-1} A_i^{(j)}\right) B_{j_1}^{(j)}, \tag{19}$$

$$v_n^{(j)} = \left(\prod_{j_1=0}^{n-1} C_{j_1}^{(j)}\right) v_0^{(j)} + \sum_{j_1=0}^{n-1}\left(\prod_{i=j_1+1}^{n-1} C_i^{(j)}\right) D_{j_1}^{(j)}, \tag{20}$$

$$w_n^{(j)} = \left(\prod_{j_1=0}^{n-1} E_{j_1}^{(j)}\right) w_0^{(j)} + \sum_{j_1=0}^{n-1}\left(\prod_{i=j_1+1}^{n-1} E_i^{(j)}\right) F_{j_1}^{(j)}, \tag{21}$$

for $n \in \mathbb{N}_0$, $j \in \{0,1,2\}$. Then, from (13)-(15) we obtain

$$\begin{aligned} u_{3n+j} &= \left(\prod_{j_1=0}^{n-1}\left(a_{3j_1+j+2}e_{3j_1+j+1}c_{3j_1+j}\right)\right)u_j \\ &+ \sum_{j_1=0}^{n-1}\left(\prod_{i=j_1+1}^{n-1}\left(a_{3i+j+2}e_{3i+j+1}c_{3i+j}\right)\right) \\ &\times \left(a_{3j_1+j+2}e_{3j_1+j+1}d_{3j_1+j}+a_{3j_1+j+2}f_{3j_1+j+1}+b_{3j_1+j+2}\right), \end{aligned} \tag{22}$$

$$\begin{aligned} v_{3n+j} &= \left(\prod_{j_1=0}^{n-1}\left(c_{3j_1+j+2}a_{3j_1+j+1}e_{3j_1+j}\right)\right)v_j \\ &+ \sum_{j_1=0}^{n-1}\left(\prod_{i=j_1+1}^{n-1}\left(c_{3i+j+2}a_{3i+j+1}e_{3i+j}\right)\right) \\ &\times \left(c_{3j_1+j+2}a_{3j_1+j+1}f_{3j_1+j}+c_{3j_1+j+2}b_{3j_1+j+1}+d_{3j_1+j+2}\right), \end{aligned} \tag{23}$$

$$\begin{aligned} w_{3n+j} &= \left(\prod_{j_1=0}^{n-1}\left(e_{3j_1+j+2}c_{3j_1+j+1}a_{3j_1+j}\right)\right)w_j \\ &+ \sum_{j_1=0}^{n-1}\left(\prod_{i=j_1+1}^{n-1}\left(e_{3i+j+2}c_{3i+j+1}a_{3i+j}\right)\right) \\ &\times \left(e_{3j_1+j+2}c_{3j_1+j+1}b_{3j_1+j}+e_{3j_1+j+2}d_{3j_1+j+1}+f_{3j_1+j+2}\right). \quad (24) \end{aligned}$$

When the coefficients are constants i.e., $a_n = a$, $b_n = b$, $c_n = c$, $d_n = d$, $e_n = e$ and $f_n = f$, formulae (22)-(24) becomes

$$u_{3n+j} = \begin{cases} (aec)^n u_j + \frac{1-(aec)^n}{1-aec}(aed+af+b), & aec \neq 1, \\ u_j + (aed+af+b)n, & aec = 1, \end{cases} \quad n \in \mathbb{N}_0, \quad (25)$$

$$v_{3n+j} = \begin{cases} (cae)^n v_j + \frac{1-(cae)^n}{1-cae}(caf+cb+d), & cae \neq 1, \\ v_j + (caf+cb+d)n, & cae = 1, \end{cases} \quad n \in \mathbb{N}_0, \quad (26)$$

$$w_{3n+j} = \begin{cases} (eca)^n w_j + \frac{1-(eca)^n}{1-eca}(ecb+ed+f), & eca \neq 1, \\ w_j + (ecb+ed+f)n, & eca = 1, \end{cases} \quad n \in \mathbb{N}_0, \quad (27)$$

for $j \in \{0,1,2\}$. From (5), we get

$$x_{n+3} = \frac{u_n}{u_{n+3}}x_{n-3k},\ n \in \mathbb{N}_0, \quad (28)$$

$$y_{n+3} = \frac{v_n}{v_{n+3}}y_{n-3k},\ n \in \mathbb{N}_0, \quad (29)$$

$$z_{n+3} = \frac{w_n}{w_{n+3}}z_{n-3k},\ n \in \mathbb{N}_0. \quad (30)$$

from which it follows that

$$x_{(3k+3)n+i} = x_{i-(3k+3)}\prod_{s=0}^{n}\frac{u_{(3k+3)s+i-3}}{u_{(3k+3)s+i}},\ i = \overline{3, 3k+5}, \quad (31)$$

$$y_{(3k+3)n+i} = y_{i-(3k+3)}\prod_{s=0}^{n}\frac{v_{(3k+3)s+i-3}}{v_{(3k+3)s+i}},\ i = \overline{3, 3k+5}, \quad (32)$$

$$z_{(3k+3)n+i} = z_{i-(3k+3)}\prod_{s=0}^{n}\frac{w_{(3k+3)s+i-3}}{w_{(3k+3)s+i}},\ i = \overline{3, 3k+5}, \quad (33)$$

for $n \in \mathbb{N}_0$.
Since the integer i can be written in the form $3m+j$, $j \in \{0,1,2\}$, then formulas in (31)-(33) becomes as follows

$$x_{(3k+3)n+3m+j} = x_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{u_{3((k+1)s+m-1)+j}}{u_{3((k+1)s+m)+j}},\ n \in \mathbb{N}_0, \quad (34)$$

$$y_{(3k+3)n+3m+j} = y_{3(m-k-1)+j} \prod_{s=0}^{n} \frac{v_{3((k+1)s+m-1)+j}}{v_{3((k+1)s+m)+j}}, \ n \in \mathbb{N}_0, \tag{35}$$

$$z_{(3k+3)n+3m+j} = z_{3(m-k-1)+j} \prod_{s=0}^{n} \frac{w_{3((k+1)s+m-1)+j}}{w_{3((k+1)s+m)+j}}, \ n \in \mathbb{N}_0, \tag{36}$$

where $m = \overline{1, k+1}$.
For the constant case and using (25)-(27) in (34)-(36), for $m = \overline{1, k+1}$, $j \in \{0, 1, 2\}$, $n \in \mathbb{N}_0$, we get

$$\begin{aligned} x_{(3k+3)n+3m+j} &= x_{3(m-k-1)+j} \\ \times \prod_{s=0}^{n} & \frac{(aed+af+b)\, x_j x_{j-3} \dots x_{j-3k} + (aec)^{(k+1)s+m-1} M_1}{(aed+af+b)\, x_j x_{j-3} \dots x_{j-3k} + (aec)^{(k+1)s+m} M_1}, \end{aligned} \tag{37}$$

$$\begin{aligned} y_{(3k+3)n+3m+j} &= y_{3(m-k-1)+j} \\ \times \prod_{s=0}^{n} & \frac{(caf+cb+d)\, y_j y_{j-3} \dots y_{j-3k} + (cae)^{(k+1)s+m-1} N_1}{(caf+cb+d)\, y_j y_{j-3} \dots y_{j-3k} + (cae)^{(k+1)s+m} N_1}, \end{aligned} \tag{38}$$

$$\begin{aligned} z_{(3k+3)n+3m+j} &= z_{3(m-k-1)+j} \\ \times \prod_{s=0}^{n} & \frac{(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m-1} R_1}{(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m} R_1}, \end{aligned} \tag{39}$$

where
$M_1 = (1 - aec - (aed + af + b)\, x_j x_{j-3} \dots x_{j-3k})$,
$N_1 = (1 - cae - (caf + cb + d)\, y_j y_{j-3} \dots y_{j-3k})$,
$R_1 = (1 - eca - (ecb + ed + f)\, z_j z_{j-3} \dots z_{j-3k})$, if $eca \neq 1$, and

$$x_{(3k+3)n+3m+j} = x_{3(m-k-1)+j} \prod_{s=0}^{n} \frac{M_2\,((k+1)\,s+m-1)+1}{M_2\,((k+1)\,s+m)+1}, \tag{40}$$

$$y_{(3k+3)n+3m+j} = y_{3(m-k-1)+j} \prod_{s=0}^{n} \frac{N_2\,((k+1)\,s+m-1)+1}{N_2\,((k+1)\,s+m)+1}, \tag{41}$$

$$z_{(3k+3)n+3m+j} = z_{3(m-k-1)+j} \prod_{s=0}^{n} \frac{R_2\,((k+1)\,s+m-1)+1}{R_2\,((k+1)\,s+m)+1}, \tag{42}$$

where $M_2 = x_j x_{j-3} \dots x_{j-3k}\,(aed + af + b)$, $N_2 = y_j y_{j-3} \dots y_{j-3k}\,(caf + cb + d)$, $R_2 = z_j z_{j-3} \dots z_{j-3k}\,(ecb + ed + f)$, if $eca = 1$.

In summary, the solutions of the system (1) with variable coefficients are given by formulas (34)-(36), where the sequences $(u_n)_{n\in\mathbb{N}_0}$, $(v_n)_{n\in\mathbb{N}_0}$ and $(w_n)_{n\in\mathbb{N}_0}$ are defined by formulas (22)-(24). However, for the constant case formulas (37)-(42) describes explicitly the form of the solutions.

In the following result we describe the set of initial values for which the solutions are not defined.

Theorem 2.1. *Assume that $a_n \neq 0$, $b_n \neq 0$, $c_n \neq 0$, $d_n \neq 0$, $e_n \neq 0$, $f_n \neq 0$, $n \in \mathbb{N}_0$. Then the forbidden set of the initial values for system (1) is the union of the two sets*

$$\left\{ \overrightarrow{X} : x_{-j} = 0 \text{ or } y_{-j} = 0 \text{ or } z_{-j} = 0,\ j = \overline{0, 3k} \right\}$$

and

$$\begin{aligned}
&\bigcup_{m \in \mathbb{N}_0} \left\{ \prod_{j=0}^{k} z_{n-3j} = \frac{1}{\alpha_m} \text{ or } \prod_{j=0}^{k} x_{n-3j} = \frac{1}{\beta_m} \text{ or } \prod_{j=0}^{k} y_{n-3j} = \frac{1}{\gamma_m},\ \text{where} \right.\\
\alpha_m \quad &:= \quad -\sum_{j=0}^{m} \left(\frac{f_{3j+i-1} + e_{3j+i-1} d_{3j+i-2} + e_{3j+i-1} c_{3j+i-2} b_{3j+i-3}}{e_{3j+i-1} c_{3j+i-2} a_{3j+i-3}} \right)\\
&\times \quad \prod_{l=0}^{j-1} \frac{1}{e_{3l+i-1} c_{3l+i-2} a_{3l+i-3}} \neq 0,\\
\beta_m \quad &:= \quad -\sum_{j=0}^{m} \left(\frac{b_{3j+i-1} + a_{3j+i-1} f_{3j+i-2} + a_{3j+i-1} e_{3j+i-2} d_{3j+i-3}}{a_{3j+i-1} e_{3j+i-2} c_{3j+i-3}} \right)\\
&\times \quad \prod_{l=0}^{j-1} \frac{1}{a_{3l+i-1} e_{3l+i-2} c_{3l+i-3}} \neq 0,\\
\gamma_m \quad &:= \quad -\sum_{j=0}^{m} \left(\frac{d_{3j+i-1} + c_{3j+i-1} b_{3j+i-2} + c_{3j+i-1} a_{3j+i-2} f_{3j+i-3}}{c_{3j+i-1} a_{3j+i-2} e_{3j+i-3}} \right)\\
&\times \quad \left. \prod_{l=0}^{j-1} \frac{1}{c_{3l+i-1} a_{3l+i-2} e_{3l+i-3}} \neq 0 \right\}. \qquad (43)
\end{aligned}$$

Proof. Let $(x_n, y_n, z_n)_{n \geq -3k}$ be a solution of system (1). If $x_{-j} = 0$ or $y_{-j} = 0$ or $z_{-j} = 0$ for some $j = \overline{0, 3k}$, then x_n, y_n, z_n can not be calculated. For example, Suppose that $z_{-j} = 0$ for some $j = \overline{0, 3k}$. We have the following cases:

(a) $j \in \{0, 3, ..., 3k\}$, in this case we get

$$x_1 = \frac{z_0 z_{-3} ... z_{-3k}}{x_{-2} ... x_{-3k+1} \left(a_0 + b_0 z_0 z_{-3} ... z_{-3k}\right)},$$

clearly x_1 will be not defined if $x_{-2} ... x_{-3k+1} \left(a_0 + b_0 z_0 z_{-3} ... z_{-3k}\right) = 0$ or $x_1 = 0$. If $x_1 = 0$, then

$$x_4 = \frac{z_3 z_0 ... z_{-3k+3}}{x_1 ... x_{-3k+4} \left(a_3 + b_3 z_3 z_0 ... z_{-3k+3}\right)}$$

will be not defined (division by $x_1 = 0$).

(b) $j \in \{2, 5, ..., 3k-1\}$, in this case we get

$$x_2 = \frac{z_1 z_{-2} ... z_{-3k+1}}{x_{-1} ... x_{-3k+2} \left(a_1 + b_1 z_1 z_{-2} ... z_{-3k+1}\right)},$$

clearly x_2 will be not defined if $x_{-1}...x_{-3k+2}\left(a_1+b_1z_1z_{-2}...z_{-3k+1}\right)=0$ or $x_2=0$. If $x_2=0$, then

$$x_5=\frac{z_4z_1...z_{-3k+4}}{x_2...x_{-3k+5}\left(a_4+b_4z_4z_1...z_{-3k+4}\right)}$$

will be not defined (division by $x_2=0$).

(c) $j\in\{1,4,...,3k-2\}$, in this case we get

$$x_3=\frac{z_2z_{-1}...z_{-3k+2}}{x_0...x_{-3k+3}\left(a_2+b_2z_2z_{-1}...z_{-3k+2}\right)},$$

clearly x_3 will be not defined if $x_0...x_{-3k+3}\left(a_2+b_2z_2z_{-1}...z_{-3k+2}\right)=0$ or $x_3=0$. If $x_3=0$, then

$$x_6=\frac{z_5z_2...z_{-3k+5}}{x_3...x_{-3k+6}\left(a_5+b_5z_5z_2...z_{-3k+5}\right)}$$

will be not defined (division by $x_3=0$).

So, we incorporate the set

$$\left\{\overrightarrow{X}:x_{-j}=0\text{ or }y_{-j}=0,\text{ or }z_{-j}=0,\ j=\overline{0,3k}\right\}$$

in the forbidden set. Now, we suppose that $x_n\neq 0$, $y_n\neq 0$ and $z_n\neq 0$. The solution $(x_n,y_n,z_n)_{n\geq -3k}$ of system (1) is not defined if and only if $a_n+b_n\prod_{j=0}^{k}z_{n-3j}=0$ or $c_n+d_n\prod_{j=0}^{k}x_{n-3j}=0$ or $e_n+f_n\prod_{j=0}^{k}y_{n-3j}=0$ that is, $\frac{1}{\prod_{j=0}^{k}z_{n-3j}}=-\frac{b_n}{a_n}$ or $\frac{1}{\prod_{j=0}^{k}x_{n-3j}}=-\frac{d_n}{c_n}$ or $\frac{1}{\prod_{j=0}^{k}y_{n-3j}}=-\frac{f_n}{e_n}$, for some $n\in\mathbb{N}_0$, are satisfied(Here we consider that $a_n\neq 0$, $c_n\neq 0$ and $e_n\neq 0$ for every $n\in\mathbb{N}_0$). From this and the substitution $u_n=\frac{1}{\prod_{j=0}^{k}x_{n-3j}}$, $v_n=\frac{1}{\prod_{j=0}^{k}y_{n-3j}}$, $w_n=\frac{1}{\prod_{j=0}^{k}z_{n-3j}}$, we get

$$u_{3m+i}=-\frac{d_{3m+i}}{c_{3m+i}},\ v_{3m+i}=-\frac{f_{3m+i}}{e_{3m+i}},\ w_{km+i}=-\frac{b_{3m+i}}{a_{3m+i}}, \tag{44}$$

for some $m\in\mathbb{N}_0$ and $i\in\{0,1,2\}$. Hence, we can determine the forbidden set of the initial values for system (1) by using the substitution $u_n=\frac{1}{\prod_{j=0}^{k}x_{n-3j}}$, $v_n=\frac{1}{\prod_{j=0}^{k}y_{n-3j}}$, $w_n=\frac{1}{\prod_{j=0}^{k}z_{n-3j}}$. Now, we consider the functions

$$\begin{aligned} f_{3m+i}(t) &:= a_{3m+i}t+b_{3m+i},\\ h_{3m+i}(t) &:= c_{3m+i}t+d_{3m+i},\\ g_{3m+i}(t) &:= e_{3m+i}t+f_{3m+i}, \end{aligned} \tag{45}$$

for $m\in\mathbb{N}_0,\ i\in\{0,1,2\}$, which correspond to the equations of (6). From (44) and (45), we can write

$$u_{3m+i}=f_{3m+i-1}\circ g_{3m+i-2}\circ h_{3(m-1)+i}\cdots\circ f_{i-1}\circ g_{i-2}\circ h_{i-3}\left(u_{i-3}\right), \tag{46}$$

$$v_{3m+i}=h_{3m+i-1}\circ f_{3m+i-2}\circ g_{3(m-1)+i}\cdots\circ h_{i-1}\circ f_{i-2}\circ g_{i-3}\left(v_{i-3}\right), \tag{47}$$

$$w_{3m+i} = g_{3m+i-1} \circ h_{3m+i-2} \circ f_{3(m-1)+i} \cdots \circ g_{i-1} \circ h_{i-2} \circ f_{i-3}\left(w_{i-3}\right), \quad (48)$$

where $m \in \mathbb{N}_0$, and $i \in \{3,4,5\}$. By using (44) and implicit forms (46)-(48) and considering
$f^{-1}_{3m+i}(0) = -\frac{b_{3m+i}}{a_{3m+i}}$, $h^{-1}_{3m+i}(0) = -\frac{d_{3m+i}}{c_{3m+i}}$, $g^{-1}_{3m+i}(0) = -\frac{f_{3m+i}}{e_{3m+i}}$, for $m \in \mathbb{N}_0$ and $i \in \{3,4,5\}$, we have

$$u_{i-3} = h^{-1}_{i-3} \circ g^{-1}_{i-2} \circ f^{-1}_{i-1} \circ \cdots \circ h^{-1}_{3(m-1)+i} \circ g^{-1}_{3m+i-2} \circ f^{-1}_{3m+i-1}(0), \quad (49)$$

$$v_{i-3} = g^{-1}_{i-3} \circ f^{-1}_{i-2} \circ h^{-1}_{i-1} \circ \cdots \circ g^{-1}_{3(m-1)+i} \circ f^{-1}_{3m+i-2} \circ h^{-1}_{3m+i-1}(0), \quad (50)$$

$$w_{i-3} = f^{-1}_{i-3} \circ h^{-1}_{i-2} \circ g^{-1}_{i-1} \circ \cdots \circ f^{-1}_{3(m-1)+i} \circ h^{-1}_{3m+i-2} \circ g^{-1}_{3m+i-1}(0), \quad (51)$$

where $f^{-1}_{3m+i}(t) = \frac{t-b_{3m+i}}{a_{3m+i}}$, $h^{-1}_{3m+i}(t) = \frac{t-d_{3m+i}}{c_{3m+i}}$, $g^{-1}_{3m+i}(t) = \frac{t-f_{3m+i}}{e_{3m+i}}$, $m \in \mathbb{N}_0$, $i \in \{3,4,5\}$. From (49)-(51), we obtain

$$\begin{aligned}
u_{i-3} &= -\sum_{j=0}^{m}\left(\frac{b_{3j+i-1} + a_{3j+i-1}f_{3j+i-2} + a_{3j+i-1}e_{3j+i-2}d_{3j+i-3}}{a_{3j+i-1}e_{3j+i-2}c_{3j+i-3}}\right) \\
&\times \prod_{l=0}^{j-1}\frac{1}{a_{3l+i-1}e_{3l+i-2}c_{3l+i-3}} \\
v_{i-3} &= -\sum_{j=0}^{m}\left(\frac{d_{3j+i-1} + c_{3j+i-1}b_{3j+i-2} + c_{3j+i-1}a_{3j+i-2}f_{3j+i-3}}{c_{3j+i-1}a_{3j+i-2}e_{3j+i-3}}\right) \\
&\times \prod_{l=0}^{j-1}\frac{1}{c_{3l+i-1}a_{3l+i-2}e_{3l+i-3}} \\
w_{i-3} &= -\sum_{j=0}^{m}\left(\frac{f_{3j+i-1} + e_{3j+i-1}d_{3j+i-2} + e_{3j+i-1}c_{3j+i-2}b_{3j+i-3}}{e_{3j+i-1}c_{3j+i-2}a_{3j+i-3}}\right) \\
&\times \prod_{l=0}^{j-1}\frac{1}{e_{3l+i-1}c_{3l+i-2}a_{3l+i-3}}
\end{aligned}$$

for some $m \in \mathbb{N}_0$ and $i \in \{3,4,5\}$. This means that if one of the conditions in (49)-(51) holds, then m-th iteration or $(m+1)$-th iteration in system (1) can not be calculated.

□

In the the following result, we show the existence of $3k+3$ periodic solutions for the system (1) with constant coefficients.

Theorem 2.2. *Assume that* $(aed+af+b)\,x_0x_{-3}...x_{-3k} = (caf+cb+d)\,y_0y_{-3}...y_{-3k} = (ecb+ed+f)\,z_0z_{-3}...z_{-3k} = 1-aec$ *and* $(1-aec)(aed+af+b)(caf+cb+d)(ecb+ed+f) \neq 0$. *Then all (well defined) solutions of system (1) are periodic with period* $3k+3$.

Proof. From the assumptions and (5), we have

$$\begin{aligned} u_0 &= \frac{1}{x_0 x_{-3} \ldots x_{-3k}} = \frac{aed+af+b}{1-aec}, \quad v_0 = \frac{1}{y_0 y_{-3} \ldots y_{-3k}} = \frac{caf+cb+d}{1-aec}, \\ w_0 &= \frac{1}{z_0 z_{-3} \ldots z_{-3k}} = \frac{ecb+ed+f}{1-aec}. \end{aligned}$$

From this and (6), it follows that

$$\begin{aligned} u_1 = aw_0 + b = a\left(\frac{ecb+ed+f}{1-aec}\right) + b = u_0, \\ v_1 = cu_0 + d = c\left(\frac{aed+af+b}{1-aec}\right) + d = v_0, \\ w_1 = ev_0 + f = e\left(\frac{caf+cb+d}{1-aec}\right) + f = w_0, \end{aligned}$$

and by induction we get,

$$\begin{aligned} u_n &= \cdots = u_0 = \frac{aed+af+b}{1-aec}, \quad v_n = \cdots = v_0 = \frac{caf+cb+d}{1-aec}, \\ w_n &= \cdots = w_0 = \frac{ecb+ed+f}{1-aec}, \ n \in \mathbb{N}_0. \end{aligned} \tag{52}$$

From (28)-(30) and (52), we get

$$x_{n+3} = x_{n-3k}, \quad y_{n+3} = y_{n-3k}, \quad z_{n+3} = z_{n-3k}, \ n \in \mathbb{N}_0,$$

that is, the solutions are periodic with period $3k+3$. □

Remark 2.1. (a): Using the change of variables (5), the following system

$$\begin{aligned} x_{n+1} &= \frac{\prod_{j=0}^{k} y_{n-3j}}{\prod_{j=1}^{k} x_{n-(3j-1)} \left(a_n + b_n \prod_{j=0}^{k} y_{n-3j}\right)}, \\ y_{n+1} &= \frac{\prod_{j=0}^{k} z_{n-3j}}{\prod_{j=1}^{k} y_{n-(3j-1)} \left(c_n + d_n \prod_{j=0}^{k} z_{n-3j}\right)}, \\ z_{n+1} &= \frac{\prod_{j=0}^{k} x_{n-3j}}{\prod_{j=1}^{k} z_{n-(3j-1)} \left(e_n + f_n \prod_{j=0}^{k} x_{n-3j}\right)}, \ n \in \mathbb{N}_0, \end{aligned}$$

can be solved, prior minor changes in the coefficients in the corresponding linear system, in the same manner as in solving system (1).

(b): The solutions of the difference equation

$$x_{n+1} = \frac{\prod_{j=0}^{k} x_{n-3j}}{\prod_{j=1}^{k} x_{n-(3j-1)} \left(a_n + b_n \prod_{j=0}^{k} x_{n-3j}\right)}, \ n \in \mathbb{N}_0,$$

can be obtained from system (1) by taking $z_{-i} = y_{-i} = x_{-i}$, $i = \overline{0, 3k}$, $a_n = c_n = e_n$, $n \in \mathbb{N}_0$ and $b_n = d_n = f_n$, $n \in \mathbb{N}_0$.

3. Asymptotic Behavior of Well-Defined Solutions of System (1) with Constants Coefficients

In this section, we derive some results on asymptotic behavior of well-defined solutions of system (1) with constants coefficients.
We will use well-known asymptotic formulas as follows:

$$\begin{aligned} ln\left(1+x\right) &= x-\frac{x^2}{2}+\mathcal{O}\left(x^3\right), \\ \left(1+x\right)^{-1} &= 1-x+\mathcal{O}\left(x^2\right), \end{aligned} \tag{53}$$

for $x \to 0$, where $\mathcal{O}$ is the Landau "big-oh" symbol.

Theorem 3.1. *Assume that* $aec = 1$, $(aed + af + b) \neq 0$, $(caf + cb + d) \neq 0$ *and* $(ecb + ed + f) \neq 0$, $x_{-i}y_{-i}z_{-i} \neq 0$ *for* $i = \overline{0, 3k}$. *Then, every well-defined solution* $\left(x_n, y_n, z_n\right)_{n \geq -3k}$ *of system* (1) *converges to zero.*

Proof. By formulas (40)-(42), we get

$$\begin{aligned} & x_{(3k+3)n+3m+j} \\ = \; & x_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{M_2\left(\left(k+1\right)s+m-1\right)+1}{M_2\left(\left(k+1\right)s+m\right)+1} \\ = \; & x_{3(m-k-1)+j}\prod_{s=0}^{n}\left(1-\frac{M_2}{M_2\left(\left(k+1\right)s+m\right)+1}\right) \\ = \; & x_{3(m-k-1)+j}C_1\left(n_0\right)\prod_{s=n_0+1}^{n}\left(1-\frac{1}{\left(k+1\right)s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right) \\ = \; & x_{3(m-k-1)+j}C_1\left(n_0\right)exp\left(\sum_{s=n_0+1}^{n} ln\left(1-\frac{1}{\left(k+1\right)s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\right) \\ = \; & x_{3(m-k-1)+j}C_1\left(n_0\right)exp\left(\frac{-1}{k+1}\sum_{s=n_0+1}^{n}\left(\frac{1}{s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\right), \end{aligned} \tag{54}$$

where $C_1\left(n_0\right) = \prod_{s=0}^{n_0}\left(1-\frac{M_2}{M_2((k+1)s+m)+1}\right)$, $m = \overline{1, k+1}$ and $j \in \{0, 1, 2\}$,

$$\begin{aligned} & y_{(3k+3)n+3m+j} \\ = \; & y_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{N_2\left(\left(k+1\right)s+m-1\right)+1}{N_2\left(\left(k+1\right)s+m\right)+1} \\ = \; & y_{3(m-k-1)+j}\prod_{s=0}^{n}\left(1-\frac{N_2}{N_2\left(\left(k+1\right)s+m\right)+1}\right) \\ = \; & y_{3(m-k-1)+j}C_2\left(n_0\right)\prod_{s=n_0+1}^{n}\left(1-\frac{1}{\left(k+1\right)s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= y_{3(m-k-1)+j} C_2\left(n_0\right) exp\left(\sum_{s=n_0+1}^{n} ln\left(1-\frac{1}{(k+1)\,s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\right)\\
&= y_{3(m-k-1)+j} C_2\left(n_0\right) exp\left(\frac{-1}{k+1}\sum_{s=n_0+1}^{n}\left(\frac{1}{s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\right), \qquad (55)
\end{aligned}
$$

where $C_2\left(n_0\right)=\prod_{s=0}^{n_0}\left(1-\frac{N_2}{N_2((k+1)s+m)+1}\right)$, $m=\overline{1,k+1}$ and $j\in\{0,1,2\}$,

$$
\begin{aligned}
& z_{(3k+3)n+3m+j}\\
&= z_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{R_2\left((k+1)\,s+m-1\right)+1}{R_2\left((k+1)\,s+m\right)+1}\\
&= z_{3(m-k-1)+j}\prod_{s=0}^{n}\left(1-\frac{R_2}{R_2\left((k+1)\,s+m\right)+1}\right)\\
&= z_{3(m-k-1)+j} C_3\left(n_0\right)\prod_{s=n_0+1}^{n}\left(1-\frac{1}{(k+1)\,s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\\
&= z_{3(m-k-1)+j} C_3\left(n_0\right) exp\left(\sum_{s=n_0+1}^{n} ln\left(1-\frac{1}{(k+1)\,s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\right)\\
&= z_{3(m-k-1)+j} C_3\left(n_0\right) exp\left(\frac{-1}{k+1}\sum_{s=n_0+1}^{n}\left(\frac{1}{s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\right), \qquad (56)
\end{aligned}
$$

where $C_3\left(n_0\right)=\prod_{s=0}^{n_0}\left(1-\frac{R_2}{R_2((k+1)s+m)+1}\right)$, $m=\overline{1,k+1}$ and $j\in\{0,1,2\}$.

Letting $n\to\infty$ in (54)-(56), using the fact that $\sum_{s=n_0+1}^{n}\frac{1}{s}\to\infty$ as $n\to\infty$ and that the series $\sum_{s=n_0+1}^{\infty}\mathcal{O}\left(\frac{1}{s^2}\right)$ converges to zero. Therefore, this result can be seen easily from (54)-(56).

□

Theorem 3.2. *Assume that* $|aec|<1$, $bdf\neq 0$, $x_j x_{j-3}\dots x_{j-3k}\neq\frac{1-aec}{aed+af+b}$, $y_j y_{j-3}\dots y_{j-3k}\neq\frac{1-aec}{caf+cb+d}$, $z_j z_{j-3}\dots z_{j-3k}\neq\frac{1-aec}{ecb+ed+f}$, $x_{-i}y_{-i}z_{-i}\neq 0$ *for* $i=\overline{0,3k}$. *Then, every well-defined solution* $(x_n,y_n,z_n)_{n\geq -3k}$ *of system* (1) *converges to a not necessarily* $(3k+3)$*-periodic solution of the system.*

Proof. We know that in this case well-defined solutions of the system are given by formulas (37)-(39). By using these formulas and asymptotic formulas (53) we have that for sufficiently large n_1

$$
\begin{aligned}
& x_{(3k+3)n+3m+j} = x_{3(m-k-1)+j}\\
\times\ & \prod_{s=0}^{n}\frac{(aed+af+b)\,x_j x_{j-3}\dots x_{j-3k}+(aec)^{(k+1)s+m-1}M_1}{(aed+af+b)\,x_j x_{j-3}\dots x_{j-3k}+(aec)^{(k+1)s+m}M_1}\\
= \ & x_{3(m-k-1)+j} C_1\left(n_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{s=n_1+1}^{n}\left(1+\frac{(aec)^{(k+1)s+m}(1-aec)M_1}{(aed+af+b)x_jx_{j-3}\dots x_{j-3k}}+\mathcal{O}\left((aec)^{2ks}\right)\right)\\
&= x_{3(m-k-1)+j}C_1(n_1)\\
&\times exp\Bigg((1-aec)\frac{M_1}{(aed+af+b)x_jx_{j-3}\dots x_{j-3k}}\\
&\times \sum_{s=n_1+1}^{n}\left((aec)^{(k+1)s+m}+\mathcal{O}\left((aec)^{2ks}\right)\right)\Bigg) \qquad (57)
\end{aligned}
$$

where $C_1(n_1)=\prod_{s=0}^{n_1}\frac{(aed+af+b)x_jx_{j-3}\dots x_{j-3k}+(aec)^{(k+1)s+m-1}M_1}{(aed+af+b)x_jx_{j-3}\dots x_{j-3k}+(aec)^{(k+1)s+m}M_1}$, $m=\overline{1,k+1}$ and $j\in\{0,1,2\}$,

$$
\begin{aligned}
&y_{(3k+3)n+3m+j}=y_{3(m-k-1)+j}\\
&\times \prod_{s=0}^{n}\frac{(caf+cb+d)y_jy_{j-3}\dots y_{j-3k}+(cae)^{(k+1)s+m-1}N_1}{(caf+cb+d)y_jy_{j-3}\dots y_{j-3k}+(cae)^{(k+1)s+m}N_1}\\
&= y_{3(m-k-1)+j}C_2(n_1)\\
&\times \prod_{s=n_1+1}^{n}\left(1+\frac{(cae)^{(k+1)s+m}(1-cae)N_1}{(caf+cb+d)y_jy_{j-3}\dots y_{j-3k}}+\mathcal{O}\left((cae)^{2ks}\right)\right)\\
&= y_{3(m-k-1)+j}C_2(n_1)\\
&\times exp\Bigg((1-cae)\frac{N_1}{(caf+cb+d)y_jy_{j-3}\dots y_{j-3k}}\\
&\times \sum_{s=n_1+1}^{n}\left((cae)^{(k+1)s+m}+\mathcal{O}\left((cae)^{2ks}\right)\right)\Bigg) \qquad (58)
\end{aligned}
$$

where $C_2(n_1)=\prod_{s=0}^{n_1}\frac{(caf+cb+d)y_jy_{j-3}\dots y_{j-3k}+(cae)^{(k+1)s+m-1}N_1}{(caf+cb+d)y_jy_{j-3}\dots y_{j-3k}+(cae)^{(k+1)s+m}N_1}$, $m=\overline{1,k+1}$ and $j\in\{0,1,2\}$,

$$
\begin{aligned}
&z_{(3k+3)n+3m+j}=z_{3(m-k-1)+j}\\
&\times \prod_{s=0}^{n}\frac{(ecb+ed+f)z_jz_{j-3}\dots z_{j-3k}+(eca)^{(k+1)s+m-1}R_1}{(ecb+ed+f)z_jz_{j-3}\dots z_{j-3k}+(eca)^{(k+1)s+m}R_1}\\
&= z_{3(m-k-1)+j}C_3(n_1)\\
&\times \prod_{s=n_1+1}^{n}\left(1+\frac{(eca)^{(k+1)s+m}(1-eca)R_1}{(ecb+ed+f)z_jz_{j-3}\dots z_{j-3k}}+\mathcal{O}\left((eca)^{2ks}\right)\right)\\
&= z_{3(m-k-1)+j}C_3(n_1)\\
&\times exp\Bigg((1-eca)\frac{R_1}{(ecb+ed+f)z_jz_{j-3}\dots z_{j-3k}}
\end{aligned}
$$

$$\times \sum_{s=n_1+1}^{n} \left((eca)^{(k+1)s+m} + \mathcal{O}\left((eca)^{2ks} \right) \right) \tag{59}$$

where $C_3(n_1) = \prod_{s=0}^{n_1} \frac{(ecb+ed+f)z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m-1} R_1}{(ecb+ed+f)z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m} R_1}$, $m = \overline{1, k+1}$ and $j \in \{0, 1, 2\}$.

From (57)-(59) and since $|aec| < 1$, it easily follows that the sequences $\left(x_{(3k+3)n+3m+j}\right)_{n\in\mathbb{N}_0}$, $\left(y_{(3k+3)n+3m+j}\right)_{n\in\mathbb{N}_0}$, $\left(z_{(3k+3)n+3m+j}\right)_{n\in\mathbb{N}_0}$ are convergent for each $m = \overline{1, k+1}$ and $j \in \{0, 1, 2\}$, from which the theorem follows. $\square$

Theorem 3.3. *Assume that* $|aec| > 1$, $bdf \neq 0$, $x_j x_{j-3} \dots x_{j-3k} \neq \frac{1-aec}{aed+af+b}$, $y_j y_{j-3} \dots y_{j-3k} \neq \frac{1-aec}{caf+cb+d}$, $z_j z_{j-3} \dots z_{j-3k} \neq \frac{1-aec}{ecb+ed+f}$, $x_{-i} y_{-i} z_{-i} \neq 0$ *for* $i = \overline{0, 3k}$. *Then, every well-defined solution* $(x_n, y_n, z_n)_{n \geq -3k}$ *of system* (1) *converges to zero.*

Proof. In this case, well-defined solutions of system (1) are also given by formulas (37)-(39). Further note that for each $m = \overline{1, k+1}$ and $j \in \{0, 1, 2\}$ holds

$$\begin{aligned} lim_{s\to\infty} \quad & \frac{(aed + af + b)\, x_j x_{j-3} \dots x_{j-3k} + (aec)^{(k+1)s+m-1} M_1}{(aed + af + b)\, x_j x_{j-3} \dots x_{j-3k} + (aec)^{(k+1)s+m} M_1} \\ = \quad & \frac{1}{aec}. \end{aligned} \tag{60}$$

Now note that $\frac{1}{|aec|} < 1$, due to the assumption $|aec| > 1$. Using this fact and (60), it follows that for sufficiently large s, say $s \geq n_2$ we get

$$\begin{aligned} & \left| \frac{(aed + af + b)\, x_j x_{j-3} \dots x_{j-3k} + (aec)^{(k+1)s+m-1} M_1}{(aed + af + b)\, x_j x_{j-3} \dots x_{j-3k} + (aec)^{(k+1)s+m} M_1} \right| \\ \leq \quad & \frac{1}{2}\left(1 + \frac{1}{|aec|}\right). \end{aligned} \tag{61}$$

From this, we get

$$\begin{aligned} & \left|x_{(3k+3)n+3m+j}\right| = \left|x_{3(m-k-1)+j}\right| C_1(n_2) \\ \times \quad & \prod_{s=n_2+1}^{n} \left| \frac{(aed + af + b)\, x_j x_{j-3} \dots x_{j-3k} + (aec)^{(k+1)s+m-1} M_1}{(aed + af + b)\, x_j x_{j-3} \dots x_{j-3k} + (aec)^{(k+1)s+m} M_1} \right| \\ \leq \quad & \left|x_{3(m-k-1)+j}\right| C_1(n_2) \prod_{s=n_2+1}^{n} \left(\frac{1}{2}\left(1 + \frac{1}{|aec|}\right) \right) \\ = \quad & \left|x_{3(m-k-1)+j}\right| C_1(n_2) \left(\frac{1}{2}\left(1 + \frac{1}{|aec|}\right) \right)^{n-n_2} \to 0 \end{aligned} \tag{62}$$

as $n \to \infty$, where

$$C_1(n_2) = \prod_{s=0}^{n_2} \left| \frac{(aed + af + b)\, x_j x_{j-3} \ldots x_{j-3k} + (aec)^{(k+1)s+m-1} M_1}{(aed + af + b)\, x_j x_{j-3} \ldots x_{j-3k} + (aec)^{(k+1)s+m} M_1} \right|,$$

$$\begin{aligned} lim_{s\to\infty} \quad & \frac{(caf + cb + d)\, y_j y_{j-3} \ldots y_{j-3k} + (cae)^{(k+1)s+m-1} N_1}{(caf + cb + d)\, y_j y_{j-3} \ldots y_{j-3k} + (cae)^{(k+1)s+m} N_1} \\ = \quad & \frac{1}{cae}. \end{aligned} \tag{63}$$

Now note that $\frac{1}{|cae|} < 1$, due to the assumption $|cae| > 1$. Using this fact and (63), it follows that for sufficiently large s, say $s \geq n_2$ we get

$$\begin{aligned} & \left| \frac{(caf + cb + d)\, y_j y_{j-3} \ldots y_{j-3k} + (cae)^{(k+1)s+m-1} N_1}{(caf + cb + d)\, y_j y_{j-3} \ldots y_{j-3k} + (cae)^{(k+1)s+m} N_1} \right| \\ \leq \quad & \frac{1}{2}\left(1 + \frac{1}{|cae|}\right). \end{aligned} \tag{64}$$

From this, we get

$$\begin{aligned} & \left|y_{(3k+3)n+3m+j}\right| = \left|y_{3(m-k-1)+j}\right| C_2(n_2) \\ \times \quad & \prod_{s=n_2+1}^{n} \left| \frac{(caf + cb + d)\, y_j y_{j-3} \ldots y_{j-3k} + (cae)^{(k+1)s+m-1} N_1}{(caf + cb + d)\, y_j y_{j-3} \ldots y_{j-3k} + (cae)^{(k+1)s+m} N_1} \right| \\ \leq \quad & \left|y_{3(m-k-1)+j}\right| C_2(n_2) \prod_{s=n_2+1}^{n} \left(\frac{1}{2}\left(1 + \frac{1}{|cae|}\right)\right) \\ = \quad & \left|y_{3(m-k-1)+j}\right| C_2(n_2) \left(\frac{1}{2}\left(1 + \frac{1}{|cae|}\right)\right)^{n-n_2} \to 0 \end{aligned} \tag{65}$$

as $n \to \infty$, where

$$C_2(n_2) = \prod_{s=0}^{n_2} \left| \frac{(caf + cb + d)\, y_j y_{j-3} \ldots y_{j-3k} + (cae)^{(k+1)s+m-1} N_1}{(caf + cb + d)\, y_j y_{j-3} \ldots y_{j-3k} + (cae)^{(k+1)s+m} N_1} \right|,$$

$$\begin{aligned} lim_{s\to\infty} \quad & \frac{(ecb + ed + f)\, z_j z_{j-3} \ldots z_{j-3k} + (eca)^{(k+1)s+m-1} R_1}{(ecb + ed + f)\, z_j z_{j-3} \ldots z_{j-3k} + (eca)^{(k+1)s+m} R_1} \\ = \quad & \frac{1}{eca}. \end{aligned} \tag{66}$$

Now note that $\frac{1}{|eca|} < 1$, due to the assumption $|eca| > 1$. Using this fact and (66), it follows that for sufficiently large s, say $s \geq n_2$ we get

$$
\begin{aligned}
& \left| \frac{(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m-1}\, R_1}{(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m}\, R_1} \right| \\
\leq \quad & \frac{1}{2}\left(1+\frac{1}{|eca|}\right). \qquad (67)
\end{aligned}
$$

From this, we get

$$
\begin{aligned}
& \left|z_{(3k+3)n+3m+j}\right| = \left|z_{3(m-k-1)+j}\right| C_3\left(n_2\right) \\
\times \quad & \prod_{s=n_2+1}^{n} \left| \frac{(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m-1}\, R_1}{(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m}\, R_1} \right| \\
\leq \quad & \left|z_{3(m-k-1)+j}\right| C_3\left(n_2\right) \prod_{s=n_2+1}^{n} \left(\frac{1}{2}\left(1+\frac{1}{|eca|}\right)\right) \\
= \quad & \left|z_{3(m-k-1)+j}\right| C_3\left(n_2\right) \left(\frac{1}{2}\left(1+\frac{1}{|eca|}\right)\right)^{n-n_2} \rightarrow 0 \qquad (68)
\end{aligned}
$$

as $n \rightarrow \infty$, where

$$
C_3\left(n_2\right) = \prod_{s=0}^{n_2} \left| \frac{(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m-1}\, R_1}{(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k} + (eca)^{(k+1)s+m}\, R_1} \right|,
$$

from which the theorem follows. □

Now, we investigate the asymptotic behavior of solution of system (1) when $aec = -1$, $(aed+af+b) \neq 0$, $(caf+cb+d) \neq 0$ and $(ecb+ed+f) \neq 0$, $x_{-i}y_{-i}z_{-i} \neq 0$ for $i = \overline{0,3k}$, from (37), (38) and (39) by employing the following formulas

$$
\begin{aligned}
& x_{(3k+3)n+3m+j} = x_{3(m-k-1)+j} \\
\times \quad & \prod_{s=0}^{n} \frac{M_2 + (-1)^{(k+1)s+m-1}\left(2-(aed+af+b)\, x_j x_{j-3} \dots x_{j-3k}\right)}{M_2 + (-1)^{(k+1)s+m}\left(2-(aed+af+b)\, x_j x_{j-3} \dots x_{j-3k}\right)}, \qquad (69) \\
& y_{(3k+3)n+3m+j} = y_{3(m-k-1)+j} \\
\times \quad & \prod_{s=0}^{n} \frac{N_2 + (-1)^{(k+1)s+m-1}\left(2-(caf+cb+d)\, y_j y_{j-3} \dots y_{j-3k}\right)}{N_2 + (-1)^{(k+1)s+m}\left(2-(caf+cb+d)\, y_j y_{j-3} \dots y_{j-3k}\right)}, \qquad (70) \\
& z_{(3k+3)n+3m+j} = z_{3(m-k-1)+j} \\
\times \quad & \prod_{s=0}^{n} \frac{R_2 + (-1)^{(k+1)s+m-1}\left(2-(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k}\right)}{R_2 + (-1)^{(k+1)s+m}\left(2-(ecb+ed+f)\, z_j z_{j-3} \dots z_{j-3k}\right)}, \qquad (71)
\end{aligned}
$$

for every $n \in \mathbb{N}_0$, $m = \overline{1, k+1}$ and $j \in \{0,1,2\}$.

Theorem 3.4. *Suppose that $aec = -1$, $(aed + af + b) \neq 0$, $(caf + cb + d) \neq 0$ and $(ecb + ed + f) \neq 0$, $x_{-i}y_{-i}z_{-i} \neq 0$ for $i = \overline{0, 3k}$, $m = \overline{1, k+1}$ and $j \in \{0, 1, 2\}$. Then the following statements hold.*

(a): *If $x_j x_{j-3} \dots x_{j-3k} = \frac{2}{aed+af+b}$, then the sequence $(x_n)_{n \geq -3k}$ is $(3k+3)$-periodic.*

(b): *If $((k+1)s + m)$ is even and $|(aed + af + b) x_j x_{j-3} \dots x_{j-3k} - 1| < 1$, then $x_{(3k+3)n+3m+j} \to 0$, as $n \to \infty$.*

(c): *If $((k+1)s + m)$ is even and $|(aed + af + b) x_j x_{j-3} \dots x_{j-3k} - 1| > 1$, then $\left|x_{(3k+3)n+3m+j}\right| \to \infty$, as $n \to \infty$.*

(d): *If $((k+1)s + m)$ is even and $(aed + af + b) x_j x_{j-3} \dots x_{j-3k} - 1 = 1$, then the sequence $(x_n)_{n \geq -3k}$ is $(3k+3)$-periodic.*

(e): *If $((k+1)s + m)$ is even and $(aed + af + b) x_j x_{j-3} \dots x_{j-3k} - 1 = -1$, then the sequence $(x_n)_{n \geq -3k}$ is $(6k+6)$-periodic.*

(f): *If $((k+1)s + m)$ is odd and $\left|\frac{1}{(aed+af+b)x_j x_{j-3} \dots x_{j-3k} - 1}\right| < 1$, then $x_{(3k+3)n+3m+j} \to 0$, as $n \to \infty$.*

(g): *If $((k+1)s + m)$ is odd and $\left|\frac{1}{(aed+af+b)x_j x_{j-3} \dots x_{j-3k} - 1}\right| > 1$, then $\left|x_{(3k+3)n+3m+j}\right| \to \infty$, as $n \to \infty$.*

(h): *If $((k+1)s + m)$ is odd and $\frac{1}{(aed+af+b)x_j x_{j-3} \dots x_{j-3k} - 1} = 1$, then the sequence $(x_n)_{n \geq -3k}$ is $(3k+3)$-periodic.*

(i): *If $((k+1)s + m)$ is odd and $\frac{1}{(aed+af+b)x_j x_{j-3} \dots x_{j-3k} - 1} = -1$, then the sequence $(x_n)_{n \geq -3k}$ is $(6k+6)$-periodic.*

(j): *If $y_j y_{j-3} \dots y_{j-3k} = \frac{2}{caf+cb+d}$, then the sequence $(y_n)_{n \geq -3k}$ is $(3k+3)$-periodic.*

(k): *If $((k+1)s + m)$ is even and $|(caf + cb + d) y_j y_{j-3} \dots y_{j-3k} - 1| < 1$, then $y_{(3k+3)n+3m+j} \to 0$, as $n \to \infty$.*

(l): *If $((k+1)s + m)$ is even and $|(caf + cb + d) y_j y_{j-3} \dots y_{j-3k} - 1| > 1$, then $\left|y_{(3k+3)n+3m+j}\right| \to \infty$, as $n \to \infty$.*

(m): *If $((k+1)s + m)$ is even and $(caf + cb + d) y_j y_{j-3} \dots y_{j-3k} - 1 = 1$, then the sequence $(y_n)_{n \geq -3k}$ is $(3k+3)$-periodic.*

(n): *If $((k+1)s + m)$ is even and $(caf + cb + d) y_j y_{j-3} \dots y_{j-3k} - 1 = -1$, then the sequence $(y_n)_{n \geq -3k}$ is $(6k+6)$-periodic.*

(o): *If $((k+1)s + m)$ is odd and $\left|\frac{1}{(caf+cb+d)y_j y_{j-3} \dots y_{j-3k} - 1}\right| < 1$, then $y_{(3k+3)n+3m+j} \to 0$, as $n \to \infty$.*

(p): *If $((k+1)s + m)$ is odd and $\left|\frac{1}{(caf+cb+d)y_j y_{j-3} \dots y_{j-3k} - 1}\right| > 1$, then $\left|y_{(3k+3)n+3m+j}\right| \to \infty$, as $n \to \infty$.*

(q): *If $((k+1)s + m)$ is odd and $\frac{1}{(caf+cb+d)y_j y_{j-3} \dots y_{j-3k} - 1} = 1$, then the sequence $(y_n)_{n \geq -3k}$ is $(3k+3)$-periodic.*

(r): *If $((k+1)s + m)$ is odd and $\frac{1}{(caf+cb+d)y_j y_{j-3} \dots y_{j-3k} - 1} = -1$, then the sequence $(y_n)_{n \geq -3k}$ is $(6k+6)$-periodic.*

(s): *If* $z_j z_{j-3} \dots z_{j-3k} = \frac{2}{ecb+ed+f}$*, then the sequence* $(z_n)_{n\geq -3k}$ *is* $(3k+3)$*-periodic.*

(t): *If* $((k+1)s+m)$ *is even and* $|(ecb+ed+f) z_j z_{j-3} \dots z_{j-3k} - 1| < 1$*, then* $z_{(3k+3)n+3m+j} \to 0$*, as* $n \to \infty$*.*

(u): *If* $((k+1)s+m)$ *is even and* $|(ecb+ed+f) z_j z_{j-3} \dots z_{j-3k} - 1| > 1$*, then* $|z_{(3k+3)n+3m+j}| \to \infty$*, as* $n \to \infty$*.*

(v): *If* $((k+1)s+m)$ *is even and* $(ecb+ed+f) z_j z_{j-3} \dots z_{j-3k} - 1 = 1$*, then the sequence* $(z_n)_{n\geq -3k}$ *is* $(3k+3)$*-periodic.*

(w): *If* $((k+1)s+m)$ *is even and* $(ecb+ed+f) z_j z_{j-3} \dots z_{j-3k} - 1 = -1$*, then the sequence* $(z_n)_{n\geq -3k}$ *is* $(6k+6)$*-periodic.*

(x): *If* $((k+1)s+m)$ *is odd and* $\left|\frac{1}{(ecb+ed+f) z_j z_{j-3} \dots z_{j-3k} - 1}\right| < 1$*, then* $z_{(3k+3)n+3m+j} \to 0$*, as* $n \to \infty$*.*

(y): *If* $((k+1)s+m)$ *is odd and* $\left|\frac{1}{(ecb+ed+f) z_j z_{j-3} \dots z_{j-3k} - 1}\right| > 1$*, then* $|z_{(3k+3)n+3m+j}| \to \infty$*, as* $n \to \infty$*.*

(z): *If* $((k+1)s+m)$ *is odd and* $\frac{1}{(ecb+ed+f) z_j z_{j-3} \dots z_{j-3k} - 1} = 1$*, then the sequence* $(z_n)_{n\geq -3k}$ *is* $(3k+3)$*-periodic.*

(z'): *If* $((k+1)s+m)$ *is odd and* $\frac{1}{(ecb+ed+f) x_j x_{j-3} \dots x_{j-3k} - 1} = -1$*, then the sequence* $(z_n)_{n\geq -3k}$ *is* $(6k+6)$*-periodic.*

Proof. Here, we will prove the items (a)-(i) since (j)-(r) and (s)-(z') can be proved similarly and are omitted.

(a): This result can be seen easily from the assumption $x_j x_{j-3} \dots x_{j-3k} = \frac{2}{aed+af+b}$ and some simple calculation from equation (69).

(b)-(e): Assume that $((k+1)s+m)$ is even. From equation (69) we get

$$\begin{aligned} & x_{(3k+3)n+3m+j} = x_{3(m-k-1)+j} \\ \times & \prod_{s=0}^{n} \frac{M_2 + (-1)^{(k+1)s+m-1} (2 - (aed+af+b) x_j x_{j-3} \dots x_{j-3k})}{M_2 + (-1)^{(k+1)s+m} (2 - (aed+af+b) x_j x_{j-3} \dots x_{j-3k})} \\ = & x_{3(m-k-1)+j} ((aed+af+b) x_j x_{j-3} \dots x_{j-3k} - 1)^{n+1} \end{aligned} \tag{72}$$

From (72), the results can be seen easily.

(f)-(i): Assume that $((k+1)s+m)$ is odd. From equation (69) we get

$$\begin{aligned} & x_{(3k+3)n+3m+j} = x_{3(m-k-1)+j} \\ \times & \prod_{s=0}^{n} \frac{M_2 + (-1)^{(k+1)s+m-1} (2 - (aed+af+b) x_j x_{j-3} \dots x_{j-3k})}{M_2 + (-1)^{(k+1)s+m} (2 - (aed+af+b) x_j x_{j-3} \dots x_{j-3k})} \\ = & x_{3(m-k-1)+j} \left(\frac{1}{(aed+af+b) x_j x_{j-3} \dots x_{j-3k} - 1}\right)^{n+1} \end{aligned} \tag{73}$$

From (73), the results can be seen easily.

□

Theorem 3.5. *Assume that $a=c=e=0$ or $x_j x_{j-3}\dots x_{j-3k}=\frac{1-aec}{aed+af+b}$, $y_j y_{j-3}\dots y_{j-3k}=\frac{1-aec}{caf+cb+d}$, $z_j z_{j-3}\dots z_{j-3k}=\frac{1-aec}{ecb+ed+f}$, $x_{-i}y_{-i}z_{-i}\neq 0$ for $i=\overline{0,3k}$. Then, every well-defined solution $(x_n,y_n,z_n)_{n\geq -3k}$ of system (1) converges to a not necessarily $(3k+3)$-periodic solution of the system.*

Proof. By formulas (37)-(39), we have

$$\begin{aligned} x_{(3k+3)n+3m+j} &= x_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{(aed+af+b)\,x_j x_{j-3}\dots x_{j-3k}}{(aed+af+b)\,x_j x_{j-3}\dots x_{j-3k}} \\ &= x_{3(m-k-1)+j},\ n\in\mathbb{N}_0, \end{aligned} \tag{74}$$

$$\begin{aligned} y_{(3k+3)n+3m+j} &= y_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{(caf+cb+d)\,y_j y_{j-3}\dots y_{j-3k}}{(caf+cb+d)\,y_j y_{j-3}\dots y_{j-3k}} \\ &= y_{3(m-k-1)+j},\ n\in\mathbb{N}_0, \end{aligned} \tag{75}$$

$$\begin{aligned} z_{(3k+3)n+3m+j} &= z_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{(ecb+ed+f)\,z_j z_{j-3}\dots z_{j-3k}}{(ecb+ed+f)\,z_j z_{j-3}\dots z_{j-3k}} \\ &= z_{3(m-k-1)+j},\ n\in\mathbb{N}_0, \end{aligned} \tag{76}$$

for each $m=\overline{1,k+1}$ and $j\in\{0,1,2\}$, Proof of the theorem can be seen easily from (74)-(76). $\square$

Finally we investigate the asymptotic behavior of solution of equations (37)-(42) when $aec\neq 0$, $b=d=f=0$, for each $m=\overline{1,k+1}$ and $j\in\{0,1,2\}$, by employing the following formulas, for the case $aec\neq 1$,

$$x_{(3k+3)n+3m+j}=x_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{1}{aec},\ n\in\mathbb{N}_0, \tag{77}$$

$$y_{(3k+3)n+3m+j}=y_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{1}{cae},\ n\in\mathbb{N}_0, \tag{78}$$

$$z_{(3k+3)n+3m+j}=z_{3(m-k-1)+j}\prod_{s=0}^{n}\frac{1}{eca},\ n\in\mathbb{N}_0, \tag{79}$$

while for $aec=1$,

$$x_{(3k+3)n+3m+j}=x_{3(m-k-1)+j},\ n\in\mathbb{N}_0, \tag{80}$$

$$y_{(3k+3)n+3m+j}=y_{3(m-k-1)+j},\ n\in\mathbb{N}_0, \tag{81}$$

$$z_{(3k+3)n+3m+j}=z_{3(m-k-1)+j},\ n\in\mathbb{N}_0, \tag{82}$$

By using above formulas, we give the following theorem. Proof of the theorem can be seen easily from (77)-(82).

Theorem 3.6. *Suppose that* $aec \neq 0$, $b = d = f = 0$, *for each* $m = \overline{1, k+1}$ *and* $j \in \{0, 1, 2\}$. *Then the next statements hold.*

(a): *If* $|aec| > 1$, *then* $x_n \to 0$, $y_n \to 0$, $z_n \to 0$, *as* $n \to \infty$.

(b): *If* $|aec| < 1$, *then* $|x_n| \to \infty$, $|y_n| \to \infty$, $|z_n| \to \infty$, *as* $n \to \infty$.

(c): *If* $aec = 1$, *then the sequences* $(x_n)_{n \geq -3k}$, $(y_n)_{n \geq -3k}$, $(z_n)_{n \geq -3k}$, *are* $(3k+3)$*-periodic.*

(d): *If* $aec = -1$, *then the sequences* $(x_n)_{n \geq -3k}$, $(y_n)_{n \geq -3k}$, $(z_n)_{n \geq -3k}$, *are* $(6k+6)$*-periodic.*

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