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BICOMPLEX GENERALIZED $k-$HORADAM QUATERNIONS

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Abstract. This study provides a broad overview of the generalization of the various quaternions, especially in the context of its enhancing importance in the disciplines of mathematics and physics. By the help of bicomplex numbers, in this paper, we define the bicomplex generalized $k-$Horadam quaternions. Fundamental properties and mathematical preliminaries of these quaternions are outlined. Finally, we give some basic conjucation identities, generating function, the Binet formula, summation formula, matrix representation and a generalized identity, which is generalization of the well-known identities such as Catalan's identity, Cassini's identity and d'Ocagne's identity, of the bicomplex generalized $k-$Horadam quaternions in detail.

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1. INTRODUCTION

Bicomplex numbers emerge in various scientific areas such as quantum mechanics, digital signal processing, electromagnetic waves and curved structures, determination of antenna patterns, fractal structures and many related fields. Recently, several remarkable studies have been conducted related with bicomplex numbers (see [6, 11, 13, 19, 22, 24, 25, 27, 29, 30, 35]). For instance, Rochon and Tremblay, in [29], studied the bicomplex Schrödinger equations. They also mentioned that the bicomplex quantum mechanics are the generalization of both the classical and hyperbolic quantum mechanics. Kabadayi and Yayli, in [19], represented a curve by means of bicomplex numbers in a hypersurface in E^4 and then they defined the homothetic motion of this curve. Lavoie et al., in [21], determined the eigenkets and eigenvalues of the bicomplex quantum harmonic oscillator Hamiltonian. They asserted that these eigenvalues and eigenkets, first in the literature, were derived with a number system larger than $\mathbb{C}$. The bicomplex number q, which extends the complex numbers, can be defined as

$$\mathbb{C}_2 = \{q = q_1 + \mathbf{i}q_2 + \mathbf{j}q_3 + \mathbf{ij}q_4 \quad | \quad q_1, q_2, q_3, q_4 \in \mathbb{R}\}, \tag{1.1}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{ij}$ satisfy the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = -1, \quad \mathbf{ij} = \mathbf{ji}. \tag{1.2}$$

The conjugations of the bicomplex numbers are defined in [28] as:

$$q_i^\star = q_1 - \mathbf{i}q_2 + \mathbf{j}q_3 - \mathbf{ij}q_4, \quad q_j^\star = q_1 + \mathbf{i}q_2 - \mathbf{j}q_3 - \mathbf{ij}q_4$$

and

$$q_{ij}^\star = q_1 - \mathbf{i}q_2 - \mathbf{j}q_3 + \mathbf{ij}q_4$$

and the basic properties of the conjugations are as follows:

$$\left(q^\star\right)^\star = q, \quad (q_1q_2)^\star = q_2^\star q_1^\star, \quad (q_1+q_2)^\star = q_1^\star + q_2^\star, \quad (\lambda q_1)^\star = \lambda q_1^\star$$

and

$$(\lambda q_1 \pm \mu q_2)^\star = \lambda q_1^\star + \mu q_2^\star,$$

where $q_1, q_2 \in \mathbb{C}_2$ and $\lambda, \mu \in \mathbb{R}$. Furthermore, three different norms for the bicomplex numbers are given by

$$N_{q_i} = ||q \times q_i|| = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2 + 2j(q_1q_3 + q_2q_4)|},$$
$$N_{q_j} = ||q \times q_j|| = \sqrt{|q_1^2 - q_2^2 + q_3^2 - q_4^2 + 2i(q_1q_2 + q_3q_4)|},$$
$$N_{q_{ij}} = ||q \times q_{ij}|| = \sqrt{|q_1^2 + q_2^2 + q_3^2 + q_4^2 + 2ij(q_1q_4 - q_2q_3)|}.$$

Quaternions, which are a number system that extends the complex numbers, arise in quantum mechanics, physics, mathematics, computer science and related areas (see [1, 2, 5, 9, 10, 12, 14, 15, 26, 31–33]). They were first introduced by William Rowan Hamilton in 1843 [14]. In general, a quaternion q, which is member of a noncommutative division algebra, is defined by

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \quad | \quad q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \tag{1.3}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ satisfy the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{1.4}$$

Note that, although quaternions are noncommutative, the bicomplex numbers and bicomplex quaternions are commutative. The conjugate of a quaternion $\bar{q}$ is defined by

$$\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \tag{1.5}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ satisfy the rules (1.4).

The quaternions have been studied by several authors in the recent years (see [1, 2, 4, 5, 9, 10, 12, 15, 26, 31–33]). For example, Horadam, in [15], defined the Fibonacci quaternions. Motivated by Horadam's study, Halici, in [9], examine some basic properties of Fibonacci and Lucas quaternions. She also gave the generating functions, the Binet formulas and derived some sums formulas for these quaternions. Liana and Wloch [31] introduced the Jacobsthal and Jacobsthal-Lucas quaternions and they gave some properties and matrix representations of these quaternions. Tan proposed the biperiodic Fibonacci quaternions whose coefficients are the biperiodic Fibonacci numbers in [33]. Later, Tan et al. described the biperiodic

Lucas quaternions and gave the generating functions, the Binet formulas and Cassini and Catalan like identities [32]. Later, by using the bicomplex numbers, Aydın, in [1], defined the bicomplex Fibonacci and Lucas quaternions as:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + ijF_{n+3}, \tag{1.6}$$

where F_n is the nth Fibonacci number. She also studied addition, subtraction, multiplication of the bicomplex Fibonacci quaternions and then gave several properties of this quaternion. Although, she mentioned the bicomplex Lucas quaternions in Theorem 2.5 in [1], she didn't give any definition of the bicomplex Lucas quaternions.

For $n \in \mathbb{N}_0$, the Fibonacci and Lucas numbers are defined by the recurrence relations

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \tag{1.7}$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1, \tag{1.8}$$

respectively. Recently, many researchers have studied several applications and generalizations for the number sequences(see [7, 8, 16–18, 20, 36]). For further information, we specially refer to book in [20]. For example, Yazlik and Taskara, in [36], defined the generalized $k-$Horadam sequence, which is generalization of many number sequences in the literature. For $n \in \mathbb{N}_0$ and $f(k)^2 + 4g(k) > 0$, the generalized $k-$Horadam sequence defined by

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}, \quad H_{k,0} = a, \quad H_{k,1} = b. \tag{1.9}$$

Note that, the Binet formula of the generalized $k-$Horadam sequence is given by, for $n \in \mathbb{N}_0$,

$$H_{k,n} = \frac{Xr_1^n - Yr_2^n}{r_1 - r_2}, \tag{1.10}$$

where $X = b - ar_2$ and $Y = b - ar_1$.

In this paper, by analogy to generalizations of Fibonacci and Lucas quaternions explained for example in [12, 26, 32, 33], we generalize families of the Fibonacci and Lucas quaternions. Hence, the next section describes the bicomplex generalized $k-$Horadam quaternions which are both generalization of the results in [1] and they include several bicomplex quaternions which are not defined before.

2. BICOMPLEX GENERALIZED $k-$HORADAM QUATERNIONS

Definition 1. For $n \in \mathbb{N}_0$, the bicomplex generalized $k-$Horadam quaternion is defined by

$$\mathcal{H}^{Q}_{k,n} = H_{k,n} + \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3}, \tag{2.1}$$

where $H_{k,n}$ is the generalized $k-$Horadam numbers which is defined in (1.9).

TABLE 1. The bicomplex generalized $k-$Horadam quaternions

$f(k)$	$g(k)$	a	b	Bicomplex generalized $k-$Horadam quaternions $\mathcal{H}_{k,n}^{Q}=H_{k,n}+\mathbf{i}H_{k,n+1}+\mathbf{j}H_{k,n+2}+\mathbf{ij}H_{k,n+3}$ $H_{k,n}=f(k)H_{k,n-1}+g(k)H_{k,n-2}$, $H_{k,0}=a$ and $H_{k,1}=b$
1	1	0	1	Bicomplex Fibonacci quaternions [1] $Q_n=F_n+\mathbf{i}F_{n+1}+\mathbf{j}F_{n+2}+\mathbf{ij}F_{n+3}$ $F_n=F_{n-1}+F_{n-2}$, $F_0=0$ and $F_1=1$
1	1	2	1	Bicomplex Lucas quaternions $\mathcal{L}_n^{Q}=L_n+\mathbf{i}L_{n+1}+\mathbf{j}L_{n+2}+\mathbf{ij}L_{n+3}$ $L_n=L_{n-1}+L_{n-2}$, $L_0=2$ and $L_1=1$
2	k	0	1	Bicomplex $k-$Pell quaternions [3] $BC_{k,n}^{P}=P_{k,n}+\mathbf{i}P_{k,n+1}+\mathbf{j}P_{k,n+2}+\mathbf{ij}P_{k,n+3}$ $P_{k,n}=2P_{k,n-1}+kP_{k,n-2}$, $P_{k,0}=0$ and $P_{k,1}=1$
2	1	0	1	Bicomplex Pell quaternions [3] $\mathcal{P}_n^{Q}=P_n+\mathbf{i}P_{n+1}+\mathbf{j}P_{n+2}+\mathbf{ij}P_{n+3}$ $P_n=2P_{n-1}+P_{n-2}$, $P_0=0$ and $P_1=1$
2	1	2	2	Bicomplex Pell-Lucas quaternions $\mathcal{PL}_n^{Q}=Q_n+\mathbf{i}Q_{n+1}+\mathbf{j}Q_{n+2}+\mathbf{ij}Q_{n+3}$ $Q_n=2Q_{n-1}+Q_{n-2}$, $Q_0=2$ and $Q_1=2$
1	2	0	1	Bicomplex Jacobsthal quaternions $\mathcal{J}_n^{Q}=J_n+\mathbf{i}J_{n+1}+\mathbf{j}J_{n+2}+\mathbf{ij}J_{n+3}$ $J_n=J_{n-1}+2J_{n-2}$, $J_0=0$ and $J_1=1$
1	2	2	1	Bicomplex Jacobsthal-Lucas quaternions $\mathcal{JL}_n^{Q}=j_n+\mathbf{i}j_{n+1}+\mathbf{j}j_{n+2}+\mathbf{ij}j_{n+3}$ $j_n=j_{n-1}+2j_{n-2}$, $j_0=2$ and $j_1=1$

It is not difficult to see from the following table that the bicomplex generalized $k-$Horadam quaternions can be reduced into several quaternions for the special cases of $f(k), g(k), a$ and b.

For $n, m \in \mathbb{N}_0$, $\mathcal{H}_{k,n}^{Q}$ and $\mathcal{H}_{k,m}^{Q}$ be two bicomplex generalized $k-$Horadam quaternions. Thus, the addition and subtraction of these two quaternions can be given as:

$$\begin{aligned}
\mathcal{H}_{k,n}^{Q}\pm\mathcal{H}_{k,m}^{Q} &= \left(H_{k,n}+\mathbf{i}H_{k,n+1}+\mathbf{j}H_{k,n+2}+\mathbf{ij}H_{k,n+3}\right) \\
&\quad \pm\left(H_{k,m}+\mathbf{i}H_{k,m+1}+\mathbf{j}H_{k,m+2}+\mathbf{ij}H_{k,m+3}\right) \\
&= \left(H_{k,n}\pm H_{k,m}\right)+\mathbf{i}\left(H_{k,n+1}\pm H_{k,m+1}\right) \\
&\quad +\mathbf{j}\left(H_{k,n+2}\pm H_{k,m+2}\right)+\mathbf{ij}\left(H_{k,n+3}\pm H_{k,m+3}\right).
\end{aligned}$$

TABLE 2. The bicomplex generalized $k-$Horadam quaternions (continued from previous page)

k	1	0	1	Bicomplex $k-$Fibonacci quaternions [34] $Q_{F_{k,n}} = F_{k,n} + \mathbf{i}F_{k,n+1} + \mathbf{j}F_{k,n+2} + \mathbf{ij}F_{k,n+3}$ $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, $F_{k,0} = 0$ and $F_{k,1} = 1$
k	1	2	k	Bicomplex $k-$Lucas quaternions $\mathcal{L}^{Q}_{k,n} = L_{k,n} + \mathbf{i}L_{k,n+1} + \mathbf{j}L_{k,n+2} + \mathbf{ij}L_{k,n+3}$ $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$, $L_{k,0} = 2$ and $L_{k,1} = k$
p	q	a	b	Bicomplex Horadam quaternions $\mathcal{H}^{Q}_{n} = H_n + \mathbf{i}H_{n+1} + \mathbf{j}H_{n+2} + \mathbf{ij}H_{n+3}$ $H_n = pH_{n-1} + qH_{n-2}$, $H_0 = a$ and $H_1 = b$

On the other hand, the multiplication of these two quaternions can be computed as:

$$\begin{aligned}
&\mathcal{H}^{Q}_{k,n} \times \mathcal{H}^{Q}_{k,m} \\
&\quad = \left(H_{k,n} + \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3}\right) \\
&\quad\quad \times \left(H_{k,m} + \mathbf{i}H_{k,m+1} + \mathbf{j}H_{k,m+2} + \mathbf{ij}H_{k,m+3}\right) \\
&\quad = \left(H_{k,n}H_{k,m} - H_{k,n+1}H_{k,m+1} - H_{k,n+2}H_{k,m+2} + H_{k,n+3}H_{k,m+3}\right) \\
&\quad\quad + \mathbf{i}\left(H_{k,n}H_{k,m+1} + H_{k,n+1}H_{k,m} - H_{k,n+2}H_{k,m+3} - H_{k,n+3}H_{k,m+2}\right) \\
&\quad\quad + \mathbf{j}\left(H_{k,n}H_{k,m+2} + H_{k,n+2}H_{k,m} - H_{k,n+1}H_{k,m+3} - H_{k,n+3}H_{k,m+1}\right) \\
&\quad\quad + \mathbf{ij}\left(H_{k,n}H_{k,m+3} + H_{k,n+3}H_{k,m} + H_{k,n+1}H_{k,m+2} + H_{k,n+2}H_{k,m+1}\right) \\
&\quad = \mathcal{H}^{Q}_{k,m} \times \mathcal{H}^{Q}_{k,n}.
\end{aligned}$$

In addition, the conjugations of the bicomplex generalized $k-$Horadam quaternions are defined as:

$$\left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{i} = H_{k,n} - \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} - \mathbf{ij}H_{k,n+3} \tag{2.2}$$

$$\left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{j} = H_{k,n} + \mathbf{i}H_{k,n+1} - \mathbf{j}H_{k,n+2} - \mathbf{ij}H_{k,n+3} \tag{2.3}$$

$$\left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{ij} = H_{k,n} - \mathbf{i}H_{k,n+1} - \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3}. \tag{2.4}$$

Theorem 1. *For any given two bicomplex generalized $k-$Horadam quaternions $\mathcal{H}^{Q}_{k,n}$ and $\mathcal{H}^{Q}_{k,m}$, where $n, m \in \mathbb{N}_0$, the following conjugation identities hold.*

$$\left(\mathcal{H}^{Q}_{k,n}\mathcal{H}^{Q}_{k,m}\right)^{\star}_{i} = \left(\mathcal{H}^{Q}_{k,m}\right)^{\star}_{i}\left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{i} = \left(\mathcal{H}^{Q}_{k,n}\right)^{\star}_{i}\left(\mathcal{H}^{Q}_{k,m}\right)^{\star}_{i}, \tag{2.5}$$

$$\left(\mathcal{H}_{k,n}^{Q}\mathcal{H}_{k,m}^{Q}\right)_{j}^{\star}=\left(\mathcal{H}_{k,m}^{Q}\right)_{j}^{\star}\left(\mathcal{H}_{k,n}^{Q}\right)_{j}^{\star}=\left(\mathcal{H}_{k,n}^{Q}\right)_{j}^{\star}\left(\mathcal{H}_{k,m}^{Q}\right)_{j}^{\star}, \tag{2.6}$$

$$\left(\mathcal{H}_{k,n}^{Q}\mathcal{H}_{k,m}^{Q}\right)_{ij}^{\star}=\left(\mathcal{H}_{k,m}^{Q}\right)_{ij}^{\star}\left(\mathcal{H}_{k,n}^{Q}\right)_{ij}^{\star}=\left(\mathcal{H}_{k,n}^{Q}\right)_{ij}^{\star}\left(\mathcal{H}_{k,m}^{Q}\right)_{ij}^{\star}. \tag{2.7}$$

Proof. By considering the Eqs. (2.2), (2.3) and (2.4), the theorem can be proved easily. □

Theorem 2. *Let* $\left(\mathcal{H}_{k,n}^{Q}\right)_{i}^{\star}$, $\left(\mathcal{H}_{k,n}^{Q}\right)_{j}^{\star}$ *and* $\left(\mathcal{H}_{k,n}^{Q}\right)_{ij}^{\star}$ *be three conjugations of the generalized $k-$Horadam quaternion. Then we obtain the following relations*

$$\begin{aligned}\mathcal{H}_{k,n}^{Q}\left(\mathcal{H}_{k,n}^{Q}\right)_{i}^{\star}&=H_{k,n}^{2}+H_{k,n+1}^{2}-H_{k,n+2}^{2}-H_{k,n+3}^{2}\\&\quad+2j\left(H_{k,n}H_{k,n+2}+H_{k,n+1}H_{k,n+3}\right)\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathcal{H}_{k,n}^{Q}\left(\mathcal{H}_{k,n}^{Q}\right)_{j}^{\star}&=H_{k,n}^{2}-H_{k,n+1}^{2}+H_{k,n+2}^{2}-H_{k,n+3}^{2}\\&\quad+2i\left(H_{k,n}H_{k,n+1}+H_{k,n+2}H_{k,n+3}\right)\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathcal{H}_{k,n}^{Q}\left(\mathcal{H}_{k,n}^{Q}\right)_{ij}^{\star}&=H_{k,n}^{2}+H_{k,n+1}^{2}+H_{k,n+2}^{2}+H_{k,n+3}^{2}\\&\quad+2ij\left(H_{k,n}H_{k,n+3}-H_{k,n+1}H_{k,n+2}\right).\end{aligned} \tag{2.10}$$

Proof. The theorem can be proved easily by using the Eqs. (2.1), (2.2), (2.3) and (2.4). Hence we omit the proof. □

Theorem 3. *For $n\in\mathbb{N}_0$, the generalized $k-$Horadam quaternion satisfies the recurrence relation*

$$\mathcal{H}_{k,n+2}^{Q}=f(k)\mathcal{H}_{k,n+1}^{Q}+g(k)\mathcal{H}_{k,n}^{Q}. \tag{2.11}$$

Proof. By considering the right hand side of the Eq. (2.11), we get

$$\begin{aligned}&f(k)\mathcal{H}_{k,n+1}^{Q}+g(k)\mathcal{H}_{k,n}^{Q}\\&\quad=f(k)\left(H_{k,n+1}+\mathbf{i}H_{k,n+2}+\mathbf{j}H_{k,n+3}+\mathbf{ij}H_{k,n+4}\right)\\&\qquad+g(k)\left(H_{k,n}+\mathbf{i}H_{k,n+1}+\mathbf{j}H_{k,n+2}+\mathbf{ij}H_{k,n+3}\right)\\&\quad=\left(f(k)H_{k,n+1}+g(k)H_{k,n}\right)+\mathbf{i}\left(f(k)H_{k,n+2}+g(k)H_{k,n+1}\right)\\&\qquad+\mathbf{j}\left(f(k)H_{k,n+3}+g(k)H_{k,n+2}\right)+\mathbf{ij}\left(f(k)H_{k,n+4}+g(k)H_{k,n+3}\right)\\&\quad=H_{k,n+2}+\mathbf{i}H_{k,n+3}+\mathbf{j}H_{k,n+4}+\mathbf{ij}H_{k,n+5}\\&\quad=\mathcal{H}_{k,n+2}^{Q},\end{aligned}$$

which is the desired result. □

The following theorem explains the generating function of the bicomplex generalized $k-$Horadam quaternions.

Theorem 4. *For $n \in \mathbb{N}_0$, the generating function for the generalized $k-$Horadam quaternion is*

$$H(t) = \frac{\mathcal{H}^Q_{k,0} + \left(\mathcal{H}^Q_{k,1} - f(k)\mathcal{H}^Q_{k,0}\right)t}{1 - f(k)t - g(k)t^2}. \tag{2.12}$$

Proof. We use the formal power series to find the generating function of $\mathcal{H}^Q_{k,n}$. Now, we define

$$H(t) = \sum_{n=0}^{\infty} \mathcal{H}^Q_{k,n} t^n = \mathcal{H}^Q_{k,0} + \mathcal{H}^Q_{k,1} t + \sum_{n=2}^{\infty} \mathcal{H}^Q_{k,n} t^n. \tag{2.13}$$

Multiplying the Eq. (2.13) both $f(k)t$ and $g(k)t^2$, we get

$$f(k)tH(t) = \sum_{n=0}^{\infty} f(k)\mathcal{H}^Q_{k,n} t^{n+1} = f(k)\mathcal{H}^Q_{k,0} t + \sum_{n=2}^{\infty} f(k)\mathcal{H}^Q_{k,n-1} t^n \tag{2.14}$$

and

$$g(k)t^2 H(t) = \sum_{n=2}^{\infty} g(k)\mathcal{H}^Q_{k,n-2} t^n. \tag{2.15}$$

By considering the above equations and doing some basic operations, we get

$$\begin{aligned}\left(1 - f(k)t - g(k)t^2\right) H(t) &= \mathcal{H}^Q_{k,0} + \mathcal{H}^Q_{k,1} t - f(k)\mathcal{H}^Q_{k,0} t \\ &\quad + \sum_{n=2}^{\infty} \left(\underbrace{\mathcal{H}^Q_{k,n} - f(k)\mathcal{H}^Q_{k,n-1} - g(k)\mathcal{H}^Q_{k,n-2}}_{0} \right) t^n.\end{aligned}$$

Therefore, we obtain

$$H(t) = \frac{\mathcal{H}^Q_{k,0} + \left(\mathcal{H}^Q_{k,1} - f(k)\mathcal{H}^Q_{k,0}\right)t}{1 - f(k)t - g(k)t^2}, \tag{2.16}$$

which is the desired result. □

Now, we give the Binet formula for the bicomplex generalized $k-$Horadam quaternions by means of the Binet formula of the generalized $k-$Horadam numbers which is defined in the Eq. (1.10).

Theorem 5. *The Binet formula for the generalized $k-$Horadam quaternion is*

$$\mathcal{H}^Q_{k,n} = \frac{\alpha^\star X r_1^n - \beta^\star Y r_2^n}{r_1 - r_2}, \tag{2.17}$$

where $\alpha^\star = 1+\boldsymbol{i}r_1+\boldsymbol{j}r_1^2+\boldsymbol{ij}r_1^3$, $\beta^\star = 1+\boldsymbol{i}r_2+\boldsymbol{j}r_2^2+\boldsymbol{ij}r_2^3$, $X = b-ar_2$, $Y = b-ar_1$, $r_1 = \frac{f(k)+\sqrt{f(k)^2+4g(k)}}{2}$ *and* $r_2 = \frac{f(k)-\sqrt{f(k)^2+4g(k)}}{2}$.

Proof. By considering the Binet formula of the generalized $k-$Horadam numbers, we get

$$\begin{aligned}
\mathcal{H}^Q_{k,n} &= H_{k,n} + \mathbf{i}H_{k,n+1} + \mathbf{j}H_{k,n+2} + \mathbf{ij}H_{k,n+3} \\
&= \frac{Xr_1^n - Yr_2^n}{r_1 - r_2} + \mathbf{i}\frac{Xr_1^{n+1} - Yr_2^{n+1}}{r_1 - r_2} + \mathbf{j}\frac{Xr_1^{n+2} - Yr_2^{n+2}}{r_1 - r_2} \\
&\quad + \mathbf{ij}\frac{Xr_1^{n+3} - Yr_2^{n+3}}{r_1 - r_2} \\
&= \frac{Xr_1^n}{r_1 - r_2}\left(1 + \mathbf{i}r_1 + \mathbf{j}r_1^2 + \mathbf{ij}r_1^3\right) - \frac{Yr_2^n}{r_1 - r_2}\left(1 + \mathbf{i}r_2 + \mathbf{j}r_2^2 + \mathbf{ij}r_2^3\right) \\
&= \frac{1}{r_1 - r_2}\left(\alpha^\star X r_1^n - \beta^\star Y r_2^n\right),
\end{aligned}$$

where $\alpha^\star = 1+\mathbf{i}r_1+\mathbf{j}r_1^2+\mathbf{ij}r_1^3$, $\beta^\star = 1+\mathbf{i}r_2+\mathbf{j}r_2^2+\mathbf{ij}r_2^3$, $X = b-ar_2$, $Y = b-ar_1$, $r_1 = \frac{f(k)+\sqrt{f(k)^2+4g(k)}}{2}$ and $r_2 = \frac{f(k)-\sqrt{f(k)^2+4g(k)}}{2}$. Therefore the proof is completed. □

The matrix representation of the bicomplex generalized $k-$Horadam quaternions can be given in the following theorem.

Theorem 6. *Let* $n \geq 1$ *be integer. Then*

$$\begin{pmatrix} \mathcal{H}^Q_{k,2n+l} & \mathcal{H}^Q_{k,2(n-1)+l} \\ \mathcal{H}^Q_{k,2(n+1)+l} & \mathcal{H}^Q_{k,2n+l} \end{pmatrix} = \begin{pmatrix} \mathcal{H}^Q_{k,2+l} & \mathcal{H}^Q_{k,l} \\ \mathcal{H}^Q_{k,4+l} & \mathcal{H}^Q_{k,2+l} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{n-1}, \tag{2.18}$$

where $l \in \{0,1\}$.

Proof. We prove the theorem by induction on n. If $n = 1$ then the result is clear. Now we assume that, for any integer m such as $1 \leq m \leq n$,

$$\begin{pmatrix} \mathcal{H}^Q_{k,2m+l} & \mathcal{H}^Q_{k,2(m-1)+l} \\ \mathcal{H}^Q_{k,2(m+1)+l} & \mathcal{H}^Q_{k,2m+l} \end{pmatrix} = \begin{pmatrix} \mathcal{H}^Q_{k,2+l} & \mathcal{H}^Q_{k,l} \\ \mathcal{H}^Q_{k,4+l} & \mathcal{H}^Q_{k,2+l} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{m-1}.$$

Then, for $n = m+1$, we get

$$\begin{pmatrix} \mathcal{H}^Q_{k,2+l} & \mathcal{H}^Q_{k,l} \\ \mathcal{H}^Q_{k,4+l} & \mathcal{H}^Q_{k,2+l} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{m}$$

$$= \begin{pmatrix} \mathcal{H}^{Q}_{k,2+l} & \mathcal{H}^{Q}_{k,l} \\ \mathcal{H}^{Q}_{k,4+l} & \mathcal{H}^{Q}_{k,2+l} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{m-1} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{H}^{Q}_{k,2m+l} & \mathcal{H}^{Q}_{k,2(m-1)+l} \\ \mathcal{H}^{Q}_{k,2(m+1)+l} & \mathcal{H}^{Q}_{k,2m+l} \end{pmatrix} \begin{pmatrix} f(k)^2+2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{H}^{Q}_{k,2(m+1)+l} & \mathcal{H}^{Q}_{k,2m+l} \\ \mathcal{H}^{Q}_{k,2(m+2)+l} & \mathcal{H}^{Q}_{k,2(m+1)+l} \end{pmatrix},$$

where $l \in \{0,1\}$. Therefore, the proof is completed. □

Theorem 7. *For $n \in \mathbb{N}_0$, the summation formula for the generalized $k-$Horadam quaternion is*

$$\sum_{s=1}^{n} \mathcal{H}^{Q}_{k,s} = \begin{cases} \frac{\mathcal{H}^{Q}_{k,n+1}+g(k)\mathcal{H}^{Q}_{k,n}-\mathcal{H}^{Q}_{k,1}-g(k)\mathcal{H}^{Q}_{k,0}}{f(k)+g(k)-1}, & \text{if } f(k)+g(k)-1 \neq 0 \\ \frac{g(k)\mathcal{H}^{Q}_{k,n}+\mathcal{H}^{Q}_{k,1}+(n-1)[g(k)a+b](1+i+j+ij)}{1+g(k)}, & \text{if } f(k)+g(k)-1 = 0 \end{cases} \tag{2.19}$$

Proof. We prove the theorem with two cases. First we assume that $f(k)+g(k)-1 \neq 0$. By using the definition of the generalized $k-$Horadam quaternion, we have

$$\sum_{s=1}^{n} \mathcal{H}^{Q}_{k,s} = \sum_{s=1}^{n} H_{k,s} + \mathbf{i}\sum_{s=1}^{n} H_{k,s+1} + \mathbf{j}\sum_{s=1}^{n} H_{k,s+2} + \mathbf{ij}\sum_{s=1}^{n} H_{k,s+3}. \tag{2.20}$$

Now, we compute each term on the right hand side of the Eq. (2.20). By using the Eq. (12) in [23], we get

$$\sum_{s=1}^{n} H_{k,s} = \frac{H_{k,n+1}+g(k)H_{k,n}-H_{k,1}-g(k)H_{k,0}}{f(k)+g(k)-1}. \tag{2.21}$$

Moreover,

$$\sum_{s=1}^{n} H_{k,s+1}$$

$$= f(k)\sum_{s=1}^{n} H_{k,s} + g(k)\sum_{s=1}^{n} H_{k,s-1}$$
$$= f(k)\left(\sum_{s=1}^{n} H_{k,s+1} - H_{k,n+1} + H_{k,1}\right)$$
$$+ g(k)\left(\sum_{s=1}^{n} H_{k,s+1} - H_{k,n+1} - H_{k,n} + H_{k,0} + H_{k,1}\right)$$

$$= \frac{g(k)H_{k,n+1} + g(k)H_{k,n} - g(k)H_{k,0} - g(k)H_{k,1} + f(k)H_{k,n+1} - f(k)H_{k,1}}{(f(k)+g(k)-1)}$$
$$= \frac{H_{k,n+2} + g(k)H_{k,n+1} - H_{k,2} - g(k)H_{k,1}}{f(k)+g(k)-1}. \tag{2.22}$$

By doing the similar operations as in the Eq. (2.22), we obtain

$$\sum_{s=1}^{n} H_{k,s+2} = \frac{H_{k,n+3} + g(k)H_{k,n+2} - H_{k,3} - g(k)H_{k,2}}{f(k)+g(k)-1} \tag{2.23}$$

and

$$\sum_{s=1}^{n} H_{k,s+3} = \frac{H_{k,n+4} + g(k)H_{k,n+3} - H_{k,4} - g(k)H_{k,3}}{f(k)+g(k)-1}. \tag{2.24}$$

By substituting the Equations (2.21), (2.22), (2.23) and (2.24) in the Eq. (2.20), one can obtain that

$$\sum_{s=1}^{n} \mathcal{H}^{Q}_{k,s} = \frac{\mathcal{H}^{Q}_{k,n+1} + g(k)\mathcal{H}^{Q}_{k,n} - \mathcal{H}^{Q}_{k,1} - g(k)\mathcal{H}^{Q}_{k,0}}{f(k)+g(k)-1}. \tag{2.25}$$

Next we assume that $f(k) + g(k) - 1 = 0$. By using the Eq. (13) in [23], we get

$$\sum_{s=1}^{n} H_{k,s} = \frac{g(k)H_{k,n} + (n-1)[g(k)a+b] + H_{k,1}}{1+g(k)}. \tag{2.26}$$

Furthermore, we have

$$\sum_{s=1}^{n} H_{k,s+2} + g(k)\sum_{s=1}^{n} H_{k,s+1} = n[g(k)a+b]$$
$$\sum_{s=1}^{n} H_{k,s+1} + H_{k,n+2} - H_{k,2} + g(k)\sum_{s=1}^{n} H_{k,s+1} = n[g(k)a+b]$$
$$(1+g(k))\sum_{s=1}^{n} H_{k,s+1} = (n-1)[g(k)a+b] + (g(k)a+b) - H_{k,n+2} + H_{k,2}$$
$$\sum_{s=1}^{n} H_{k,s+1} = \frac{g(k)H_{k,n+1} + (n-1)[g(k)a+b] + H_{k,2}}{1+g(k)}. \tag{2.27}$$

By doing the similar operations as in the Eq. (2.27), we obtain

$$\sum_{s=1}^{n} H_{k,s+2} = \frac{g(k)H_{k,n+2} + (n-1)[g(k)a+b] + H_{k,3}}{1+g(k)} \tag{2.28}$$

and

$$\sum_{s=1}^{n} H_{k,s+3} = \frac{g(k)H_{k,n+3} + (n-1)[g(k)a+b] + H_{k,4}}{1+g(k)}. \tag{2.29}$$

Therefore, by substituting the Equations (2.26), (2.27), (2.28) and (2.29) in the Eq. (2.20), one can obtain that

$$\sum_{s=1}^{n} \mathcal{H}_{k,s}^{Q} = \frac{g(k)\mathcal{H}_{k,n}^{Q} + \mathcal{H}_{k,1}^{Q} + (n-1)[g(k)a+b](1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{ij})}{1+g(k)}. \tag{2.30}$$

□

Now, we give a theorem which generalizes the well-known identities such as Catalan's identity, Cassini's identity and d'Ocagne's identity.

Theorem 8. *The generalized $k-$Horadam quaternions satisfy the identity that is*

$$\begin{aligned}
&\mathcal{H}_{k,n}^{Q}\mathcal{H}_{k,n-r+s}^{Q} - \mathcal{H}_{k,n+s}^{Q}\mathcal{H}_{k,n-r}^{Q} \\
&\quad= \frac{(-g(k))^{n-r}\left(bH_{k,r}-aH_{k,r+1}\right)\left(bH_{k,s}-aH_{k,s+1}\right)}{b^2-a^2g(k)-abf(k)} \\
&\qquad\times\Bigg[1+g(k)-g(k)^2-g(k)^3+\boldsymbol{i}\left(1-g(k)^2\right)f(k) \\
&\qquad\quad+\boldsymbol{j}(1+g(k))\left(f(k)^2+2g(k)\right)+\boldsymbol{ij}f(k)\left(f(k)^2+2g(k)\right)\Bigg].
\end{aligned} \tag{2.31}$$

Proof. By using the Theorem 7 in [36] and definition of the generalized $k-$Horadam numbers, we get

$$\begin{aligned}
&\mathcal{H}_{k,n}^{Q}\mathcal{H}_{k,n-r+s}^{Q} - \mathcal{H}_{k,n+s}^{Q}\mathcal{H}_{k,n-r}^{Q} \\
&\quad= \left(H_{k,n}+\mathbf{i}H_{k,n+1}+\mathbf{j}H_{k,n+2}+\mathbf{ij}H_{k,n+3}\right) \\
&\qquad\times\left(H_{k,n-r+s}+\mathbf{i}H_{k,n-r+s+1}+\mathbf{j}H_{k,n-r+s+2}+\mathbf{ij}H_{k,n-r+s+3}\right) \\
&\qquad-\left(H_{k,n+s}+\mathbf{i}H_{k,n+s+1}+\mathbf{j}H_{k,n+s+2}+\mathbf{ij}H_{k,n+s+3}\right) \\
&\qquad\times\left(H_{k,n-r}+\mathbf{i}H_{k,n-r+1}+\mathbf{j}H_{k,n-r+2}+\mathbf{ij}H_{k,n-r+3}\right) \\
&\quad= H_{k,n}H_{k,n-r+s} - H_{k,n+1}H_{k,n-r+s+1} - H_{k,n+2}H_{k,n-r+s+2} \\
&\qquad+ H_{k,n+3}H_{k,n-r+s+3} - H_{k,n+s}H_{k,n-r} + H_{k,n+s+1}H_{k,n-r+1} \\
&\qquad+ H_{k,n+s+2}H_{k,n-r+2} - H_{k,n+s+3}H_{k,n-r+3} \\
&\qquad+\mathbf{i}\Big(H_{k,n+1}H_{k,n-r+s} + H_{k,n}H_{k,n-r+s+1}
\end{aligned}$$

$$
\begin{aligned}
&\quad - H_{k,n+2}H_{k,n-r+s+3} - H_{k,n+3}H_{k,n-r+s+2} - H_{k,n+s}H_{k,n-r+1} \\
&\quad - H_{k,n+s+1}H_{k,n-r} + H_{k,n+s+2}H_{k,n-r+3} + H_{k,n+s+3}H_{k,n-r+2}\Big) \\
&+ \mathbf{j}\Big(H_{k,n+2}H_{k,n-r+s} + H_{k,n}H_{k,n-r+s+2} - H_{k,n+1}H_{k,n-r+s+3} \\
&\quad - H_{k,n+3}H_{k,n-r+s+1} - H_{k,n+s+2}H_{k,n-r} - H_{k,n+s}H_{k,n-r+2} \\
&\quad + H_{k,n+s+1}H_{k,n-r+3} + H_{k,n+s+3}H_{k,n-r+1}\Big) \\
&+ \mathbf{ij}\Big(H_{k,n}H_{k,n-r+s+3} + H_{k,n+3}H_{k,n-r+s} + H_{k,n+1}H_{k,n-r+s+2} \\
&\quad + H_{k,n+2}H_{k,n-r+s+1} - H_{k,n+s}H_{k,n-r+3} - H_{k,n+s+3}H_{k,n-r} \\
&\quad - H_{k,n+s+1}H_{k,n-r+2} - H_{k,n+s+2}H_{k,n-r+1}\Big) \\
&= \frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2g(k) - abf(k)} \\
&\quad \times \left[1 + g(k) - g(k)^2 - g(k)^3\right] \\
&+ \mathbf{i}\frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2g(k) - abf(k)} \\
&\quad \times \left[\left(1 - g(k)^2\right)f(k)\right] \\
&+ \mathbf{j}\frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2g(k) - abf(k)} \\
&\quad \times \left[(1 + g(k))\left(f(k)^2 + 2g(k)\right)\right] \\
&+ \mathbf{ij}\frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2g(k) - abf(k)} \\
&\quad \times \left[f(k)\left(f(k)^2 + 2g(k)\right)\right] \\
&= \frac{(-g(k))^{n-r}\left(bH_{k,r} - aH_{k,r+1}\right)\left(bH_{k,s} - aH_{k,s+1}\right)}{b^2 - a^2g(k) - abf(k)} \\
&\quad \times \Big[1 + g(k) - g(k)^2 - g(k)^3 + \mathbf{i}\left(1 - g(k)^2\right)f(k)
\end{aligned}
$$

$$+\mathbf{j}(1+g(k))\left(f(k)^2+2g(k)\right)+\mathbf{ij}f(k)\left(f(k)^2+2g(k)\right)\Bigg].$$

□

Corollary 1. *By taking $s=m-n+r$ in the Theorem 8, we obtain the following identity:*

$$\begin{aligned}&\mathcal{H}^{Q}_{k,n}\mathcal{H}^{Q}_{k,m}-\mathcal{H}^{Q}_{k,m+r}\mathcal{H}^{Q}_{k,n-r}\\ &\quad=\frac{(-g(k))^{n-r}\left(bH_{k,r}-aH_{k,r+1}\right)\left(bH_{k,m-n+r}-aH_{k,m-n+r+1}\right)}{b^2-a^2g(k)-abf(k)}\\ &\qquad\times\Bigg[1+g(k)-g(k)^2-g(k)^3+\boldsymbol{i}\left(1-g(k)^2\right)f(k)\\ &\qquad\qquad+\boldsymbol{j}(1+g(k))\left(f(k)^2+2g(k)\right)+\boldsymbol{ij}f(k)\left(f(k)^2+2g(k)\right)\Bigg].\end{aligned}$$

Corollary 2. *(Catalan's identity) By taking $m=n$ in Corollary 1, we obtain Catalan's identity for the generalized $k-$Horadam quaternions as:*

$$\begin{aligned}&\left(\mathcal{H}^{Q}_{k,n}\right)^2-\mathcal{H}^{Q}_{k,n+r}\mathcal{H}^{Q}_{k,n-r}\\ &\quad=\frac{(-g(k))^{n-r}\left(bH_{k,r}-aH_{k,r+1}\right)^2}{b^2-a^2g(k)-abf(k)}\\ &\qquad\times\Bigg[1+g(k)-g(k)^2-g(k)^3+\boldsymbol{i}\left(1-g(k)^2\right)f(k)\\ &\qquad\qquad+\boldsymbol{j}(1+g(k))\left(f(k)^2+2g(k)\right)+\boldsymbol{ij}f(k)\left(f(k)^2+2g(k)\right)\Bigg].\end{aligned}$$

Corollary 3. *(Cassini's identity) By taking $m=n$ and $r=1$ in Corollary 1, we obtain Cassini's identity for the generalized $k-$Horadam quaternions as:*

$$\begin{aligned}&\left(\mathcal{H}^{Q}_{k,n}\right)^2-\mathcal{H}^{Q}_{k,n+1}\mathcal{H}^{Q}_{k,n-1}\\ &\quad=(-g(k))^{n-1}\left(b^2-a^2g(k)-abf(k)\right)\\ &\qquad\times\Bigg[1+g(k)-g(k)^2-g(k)^3+\boldsymbol{i}\left(1-g(k)^2\right)f(k)\\ &\qquad\qquad+\boldsymbol{j}(1+g(k))\left(f(k)^2+2g(k)\right)+\boldsymbol{ij}f(k)\left(f(k)^2+2g(k)\right)\Bigg].\end{aligned}$$

Corollary 4. *(d'Ocagne's identity) By taking $n = n+1$ and $r = 1$ in Corollary 1, we obtain d'Ocagne's identity for the generalized $k-$Horadam quaternions as:*

$$\begin{aligned}
&\mathcal{H}^{Q}_{k,n+1}\mathcal{H}^{Q}_{k,m} - \mathcal{H}^{Q}_{k,m+1}\mathcal{H}^{Q}_{k,n} \\
&\quad = (-g(k))^{n}\left(bH_{k,m-n} - aH_{k,m-n+1}\right) \\
&\quad\quad \times \Bigg[1 + g(k) - g(k)^2 - g(k)^3 + \boldsymbol{i}\left(1 - g(k)^2\right) f(k) \\
&\quad\quad\quad + \boldsymbol{j}\left(1 + g(k)\right)\left(f(k)^2 + 2g(k)\right) + \boldsymbol{ij} f(k)\left(f(k)^2 + 2g(k)\right) \Bigg].
\end{aligned}$$

3. Conclusion

This study presents the bicomplex generalized $k-$Horadam quaternions which are generalization of the results in [1, 3, 34]. Moreover, for the special cases of $f(k), g(k), a$ and b, we obtain several new quaternions(see Table 1) which are not defined before in the literature. We derive the Binet formula, generating function, matrix representation and the summation formula for this quaternion. We also give Theorem 8 which is the generalization of the Catalan's identity, Cassini's identity and d'Ocagne's identity. Since this study includes some new generalized results for the bicomplex quaternions, it contributes to the literature by providing essential information on the generalization of the bicomplex quaternions.

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