MODIFIED GENERALIZED FIBONACCI AND LUCAS QUATERNIONS

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MODIFIED GENERALIZED FIBONACCI AND LUCAS QUATERNIONS

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Abstract. *In this paper, we introduce the modified generalized Fibonacci and Lucas quaternions. We present the generating functions, the Binet formulas and some significant identities for these quaternions. Also, we give the matrix representations of the modified generalized Fibonacci and modified generalized Lucas quaternions.*

Keywords: *Modified generalized Fibonacci sequence, modified generalized Lucas sequence, Recurrence Relations, Quaternions.*

1. INTRODUCTION

The Fibonacci numbers play an important role in various areas such as mathematics, physics, computer science and related fields. The Fibonacci numbers are defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 0. \tag{1.1}$$

Another significant number sequence is the Lucas numbers which is defined by the recurrence relation

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geq 0. \tag{1.2}$$

In recent years, there have been a lot of applications and generalizations of the Fibonacci and Lucas numbers in the literature [1-8]. For example, Falcon and Plaza[2] introduced the $k-$Fibonacci sequence, $\{F_{k,n}\}_{n=0}^{\infty}$, by examining the recursive application of two geometrical transformations used in the (4TLE) partition. Yazlik and Taskara[4] defined the generalized $k-$Horadam sequence and proved some properties of this sequence by means of determinant. In particular, Edson and Yayenie[5] presented an important generalization of the Fibonacci numbers, which is called as bi-periodic Fibonacci sequence, and then they derived the extended Binet formula, generating function and lots of identities of this sequence. Furthermore, Yayenie[6] defined the modified generalized Fibonacci sequence as

$$Q_0 = 0, \quad Q_1 = 1, \quad Q_n = \begin{cases} aQ_{n-1} + cQ_{n-2} & \text{if } n \text{ is even} \\ bQ_{n-1} + dQ_{n-2} & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2 \tag{1.3}$$

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where a, b, c and d are real numbers. Also he gave generating function, the generalized Binet formula and some basic identities for Q_n. Motivating by the studies [5] and [6], Bilgici [7] defined the bi-periodic Lucas numbers and modified generalized Lucas numbers and gave generating functions, the Binet formulas and some special identities for these sequences. He defined the modified generalized Lucas sequence as

$$U_0 = \frac{d+1}{d}, \quad U_1 = a, \quad U_n = \begin{cases} bU_{n-1} + dU_{n-2} & \text{if } n \text{ is even} \\ aU_{n-1} + cU_{n-2} & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2, \tag{1.4}$$

where a, b, c and d are real numbers. The generating functions of Q_n and U_n are given by

$$H(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{x(1+ax-cx^2)}{1-(ab+c+d)x^2+cdx^4} \tag{1.5}$$

and

$$U(x) = \sum_{n=0}^{\infty} U_n x^n = \frac{1}{d}\left(\frac{d+1+adx-(ab+cd+c)x^2+adx^3}{1-(ab+c+d)x^2+cdx^4}\right), \tag{1.6}$$

respectively. In addition, the Binet formulas of the sequences Q_n and U_n are also given by the following formulas:

$$Q_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^{\lfloor\frac{n}{2}\rfloor}(\alpha+d-c)^{n-\lfloor\frac{n}{2}\rfloor} - \beta^{\lfloor\frac{n}{2}\rfloor}(\beta+d-c)^{n-\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta}\right) \tag{1.7}$$

and

$$U_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\left(\frac{(\alpha+d+1)\alpha^{\lfloor\frac{n-1}{2}\rfloor}(\alpha+d-c)^{\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta} - \frac{(\beta+d+1)\beta^{\lfloor\frac{n-1}{2}\rfloor}(\beta+d-c)^{\lfloor\frac{n}{2}\rfloor}}{\alpha-\beta}\right) \tag{1.8}$$

where $\alpha = \frac{ab+c-d+\sqrt{(ab+c-d)^2+4abd}}{2}$ and $\beta = \frac{ab+c-d-\sqrt{(ab+c-d)^2+4abd}}{2}$ are the roots of the polynomial $x^2 - (ab+c-d)x - abd = 0$ and $\xi(n) = n - 2\lfloor\frac{n}{2}\rfloor$ is the parity function that we use throughout the paper. Note that, we also assume that $\Delta = (ab+c-d)^2 + 4abd > 0$.

The quaternions, which arise in quantum mechanics, physics, mathematics, computer science and related areas, are a number system that extends the complex numbers. The quaternions were first introduced by William Rowan Hamilton in 1843. In general, a quaternion q is defined by the following formula

$$q = q_0 + iq_1 + jq_2 + kq_3, \tag{1.9}$$

where i, j, k are standart orthonormal basis in $\mathbb{R}^3$ and q_0, q_1, q_2 and q_3 are real numbers. In addition, the basis i, j, k satisfy the following multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.10}$$

Note that, the conjugate of the quaternion $\bar{q}$ is defined by

$$\bar{q} = q_0 - iq_1 - jq_2 - kq_3, \tag{1.11}$$

where i, j, k satisfy the rules (1.10). Recently, the quaternions have attracted much attention of researchers [9-18]. For example, Horadam [9] defined the nth Fibonacci and Lucas quaternions as

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \tag{1.12}$$

and

$$P_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3} \tag{1.13}$$

respectively. Ramirez [11] studied the combinatorial properties of the $k-$Fibonacci and $k-$Lucas quaternions by using the properties of the $k-$Fibonacci and $k-$Lucas numbers. Tan [12] proposed the bi-periodic Fibonacci quaternion Q_n as

$$Q_n = \sum_{l=0}^{3} q_{n+l}e_l, \quad n \geq 0, \tag{1.14}$$

where q_n is the bi-periodic Fibonacci sequence(see [5]). Tan et al. [13] also presented the bi-periodic Lucas quaternion P_n as

$$P_n = \sum_{l=0}^{3} l_{n+l}e_l, \quad n \geq 0, \tag{1.15}$$

where l_n is the bi-periodic Lucas numbers(see [7]), and then they obtained generating functions, the Binet formulas and some identities between bi-periodic Fibonacci and the bi-periodic Lucas quaternions.

In this paper, motivated by the studies [9-18], we present the generating functions, the Binet formulas, Catalan's identity, Cassini's identity and some new results for both the modified generalized Fibonacci and modified generalized Lucas quaternions, which are generalization of several studies in the literature such as [10-15], in Section 2 and 3. In the final section, we give a concise conclusion.

2. MODIFIED GENERALIZED FIBONACCI QUATERNIONS

In this section, we define the modified generalized Fibonacci quaternion Θ_n. We give the generating function, the Binet formula and some significant identities for this quaternion.

Definition 2.1. For $n \in \mathbb{N}_0$, the modified generalized Fibonacci quaternion Θ_n is defined by

$$\Theta_n = \sum_{l=0}^{3} Q_{n+l}e_l, \tag{2.1}$$

where Q_n is the nth modified generalized Fibonacci numbers that is defined in (1.3). Moreover, Eq. (2.1) can be reduced to several quaternions for the special cases of a, b, c and d.

  For example,
  • if we take $c = d = 1$, we obtain the bi-periodic Fibonacci quaternions in [12],
  • if we take $c = d = 1$ and $a = b = k$, we obtain the k −Fibonacci quaternions in [11],
  • if we take $c = d = 1$ and $a = b = 2$, we obtain the Pell Fibonacci quaternions in [14],
  • if we take $c = d = 2$ and $a = b = 1$, we obtain the Jacobsthal quaternions in [15] and
  • if we take $c = d = 1$ and $a = b = 1$, we obtain the Fibonacci quaternions in [10].

Theorem 2.2 The generating function for the modified generalized Fibonacci quaternion Θ_n is

$$G(t) = \frac{\Theta_0 + (\Theta_1 - b\Theta_0)t + (a-b)R_1(t) + (c-d)R_2(t)}{1 - bt - dt^2}, \tag{2.2}$$

where

$$R_1(t) = e_0 t f(t) + e_1(f(t) - t) + e_2\left(\frac{f(t)}{t} - 1\right) + e_3\left(\frac{f(t) - (t + (ab+d)t^3)}{t^2}\right),$$

$$R_2(t) = e_0 t^2 h(t) + e_1 t h(t) + e_2 h(t) + e_3\left(\frac{h(t) - at^2}{t}\right),$$

$$f(t) = \frac{t - ct^3}{1 - (ab + d + c)t^2 + cdt^4}$$

and

$$h(t) = \frac{at^2}{1 - (ab + d + c)t^2 + cdt^4}.$$

Proof. We use formal power series representation in order to find the generating function of Θ_m. Now we define

$$G(t) = \sum_{m=0}^{\infty} \Theta_m t^m = \Theta_0 + \Theta_1 t + \sum_{m=2}^{\infty} \Theta_m t^m. \tag{2.3}$$

Note that,

$$btG(t) = \sum_{m=0}^{\infty} b\Theta_m t^{m+1} = \sum_{m=1}^{\infty} b\Theta_{m-1}t^m = bt\Theta_0 + \sum_{m=2}^{\infty} b\Theta_{m-1}t^m \tag{2.4}$$

and

$$dt^2 G(t) = \sum_{m=0}^{\infty} d\Theta_m t^{m+2} = \sum_{m=2}^{\infty} d\Theta_{m-2}t^m. \tag{2.5}$$

  Since Q_n satisfies the recurrence relations $Q_{2m} = aQ_{2m-1} + cQ_{2m-2}$ and $Q_{2m+1} = bQ_{2m} + dQ_{2m-1}$, we obtain

$$(1 - bt - dt^2)G(t) = \Theta_0 + (\Theta_1 - b\Theta_0)t + \sum_{m=2}^{\infty} (\Theta_m - b\Theta_{m-1} - d\Theta_{m-2})t^m$$
$$= \Theta_0 + (\Theta_1 - b\Theta_0)t$$
$$+ e_0\left((a - b)t\sum_{m=1}^{\infty} Q_{2m-1}t^{2m-1} + (c - d)t^2\sum_{m=1}^{\infty} Q_{2m-2}t^{2m-2}\right)$$
$$+ e_1\left((a - b)\sum_{m=2}^{\infty} Q_{2m-1}t^{2m-1} + (c - d)t\sum_{m=2}^{\infty} Q_{2m-2}t^{2m-2}\right)$$
$$+ e_2\left(\left(\frac{a-b}{t}\right)\sum_{m=2}^{\infty} Q_{2m-1}t^{2m-1} + (c - d)\sum_{m=2}^{\infty} Q_{2m-2}t^{2m-2}\right)$$
$$+ e_3\left(\left(\frac{a-b}{t^2}\right)\sum_{m=3}^{\infty} Q_{2m-1}t^{2m-1} + \left(\frac{c-d}{t}\right)\sum_{m=3}^{\infty} Q_{2m-2}t^{2m-2}\right)$$
$$= \Theta_0 + (\Theta_1 - b\Theta_0)t + e_0\left((a - b)tf(t) + (c - d)t^2h(t)\right)$$
$$+ e_1\left((a - b)(f(t) - t) + (c - d)th(t)\right)$$
$$+ e_2\left(\left(\frac{a-b}{t}\right)(f(t) - t) + (c - d)h(t)\right)$$
$$+ e_3\left(\left(\frac{a-b}{t^2}\right)(f(t) - t - (ab + d)t^3) + \left(\frac{c-d}{t}\right)(h(t) - at^2)\right)$$
$$= \Theta_0 + (\Theta_1 - b\Theta_0)t + (a - b)R_1(t) + (c - d)R_2(t),$$

where

$$R_1(t) = e_0 tf(t) + e_1(f(t) - t) + e_2\left(\frac{f(t)}{t} - 1\right) + e_3\left(\frac{f(t)-(t+(ab+d)t^3)}{t^2}\right),$$

$$R_2(t) = e_0 t^2 h(t) + e_1 th(t) + e_2 h(t) + e_3\left(\frac{h(t)-at^2}{t}\right),$$

$$f(t) = \sum_{m=1}^{\infty} Q_{2m-1}t^{2m-1} \quad \text{and} \quad h(t) = \sum_{m=1}^{\infty} Q_{2m-2}t^{2m-2}.$$

On the other hand, the modified generalized Fibonacci numbers satisfy

$$Q_{2m-1} = bQ_{2m-2} + dQ_{2m-3}$$
$$= b(aQ_{2m-3} + cQ_{2m-4}) + dQ_{2m-3}$$
$$= (ab + d)Q_{2m-3} + bcQ_{2m-4}$$
$$= (ab + d)Q_{2m-3} + cQ_{2m-3} - cdQ_{2m-5}$$
$$= (ab + d + c)Q_{2m-3} - cdQ_{2m-5}, \tag{2.6}$$

and

$$Q_{2m-2} = aQ_{2m-3} + cQ_{2m-4}$$
$$= a(bQ_{2m-4} + dQ_{2m-5}) + cQ_{2m-4}$$
$$= (ab + c)Q_{2m-4} + adQ_{2m-5}$$
$$= (ab + c)Q_{2m-4} + dQ_{2m-4} - cdQ_{2m-6}$$
$$= (ab + d + c)Q_{2m-4} - cdQ_{2m-6}. \tag{2.7}$$

Using (2.6) and (2.7), we obtain

$$(1 - (ab + d + c)t^2 + cdt^4)f(t)$$
$$= t + (ab + d)t^3 - (ab + d + c)t^3$$
$$+ \sum_{m=3}^{\infty} (Q_{2m-1} - (ab + d + c)Q_{2m-3} + cdQ_{2m-5})t^{2m-1},$$

and

$$(1 - (ab + d + c)t^2 + cdt^4)h(t)$$
$$= at^2 + \sum_{m=3}^{\infty} (Q_{2m-2} - (ab + d + c)Q_{2m-4} + cdQ_{2m-6})t^{2m-2}.$$

Rearranging the above expressions, we get

$$f(t) = \frac{t-ct^3}{1-(ab+d+c)t^2+cdt^4} \text{ and } h(t) = \frac{at^2}{1-(ab+d+c)t^2+cdt^4}.$$

Therefore, by using $f(t)$, $h(t)$, $R_1(t)$ and $R_2(t)$, we obtain the generating function of Θ_n as:

$$G(t) = \frac{\Theta_0 + (\Theta_1 - b\Theta_0)t + (a-b)R_1(t) + (c-d)R_2(t)}{1 - bt - dt^2}. \tag{2.8}$$

Now, we derive the Binet formula of the modified generalized Fibonacci quaternion by the help of the Binet formula of Q_n.

Theorem 2.3 For n $\in \mathbb{N}_0$, the Binet formula for the modified generalized Fibonacci quaternion is

$$\Theta_n = \frac{1}{(ab)^{\lfloor\frac{n}{2}\rfloor}} \frac{\alpha_{\xi(n)}\alpha^{\lfloor\frac{n}{2}\rfloor}(\alpha+d-c)^{n-\lfloor\frac{n}{2}\rfloor} - \beta_{\xi(n)}\beta^{\lfloor\frac{n}{2}\rfloor}(\beta+d-c)^{n-\lfloor\frac{n}{2}\rfloor}}{\alpha - \beta}, \tag{2.9}$$

where

$$\alpha_{\xi(n)} = \sum_{l=0}^3 \frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\lfloor\frac{l+\xi(n)}{2}\rfloor}}(\alpha+d-c)^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}\alpha^{\lfloor\frac{l+\xi(n)}{2}\rfloor}e_l$$

and

$$\beta_{\xi(n)} = \sum_{l=0}^3 \frac{a^{\xi(l+1-\xi(n))}}{(ab^{\lfloor\frac{l+\xi(n)}{2}\rfloor})}(\beta+d-c)^{\lfloor\frac{l+1-\xi(n)}{2}\rfloor}\beta^{\lfloor\frac{l+\xi(n)}{2}\rfloor}e_l.$$

Proof. By using the Binet formula of the modified generalized Fibonacci sequence, we can write

$$\Theta_{2n} = \sum_{l=0}^3 Q_{2n+l}e_l$$

$$= e_0 \frac{a}{(ab)^n}\left(\frac{\alpha^n(\alpha+d-c)^n - \beta^n(\beta+d-c)^n}{\alpha-\beta}\right)$$
$$+ e_1 \frac{1}{(ab)^n}\left(\frac{\alpha^n(\alpha+d-c)^{n+1} - \beta^n(\beta+d-c)^{n+1}}{\alpha-\beta}\right)$$
$$+ e_2 \frac{a}{(ab)^{n+1}}\left(\frac{\alpha^{n+1}(\alpha+d-c)^{n+1} - \beta^{n+1}(\beta+d-c)^{n+1}}{\alpha-\beta}\right)$$
$$+ e_3 \frac{1}{(ab)^{n+1}}\left(\frac{\alpha^{n+1}(\alpha+d-c)^{n+2} - \beta^{n+1}(\beta+d-c)^{n+2}}{\alpha-\beta}\right)$$

$$= \frac{1}{(ab)^n}\frac{\alpha^n(\alpha+d-c)^n}{\alpha-\beta}\left(e_0 a + e_1(\alpha+d-c) + e_2\left(\frac{a\alpha(\alpha+d-c)}{ab}\right)\right.$$
$$\left. + e_3\left(\frac{\alpha(\alpha+d-c)^2}{ab}\right)\right) - \frac{1}{(ab)^n}\frac{\beta^n(\beta+d-c)^n}{\alpha-\beta} \times (e_0 a +$$

$$+e_1(\beta + d - c) + e_2\left(\frac{a\beta(\beta+d-c)}{ab}\right) + e_3\left(\frac{\beta(\beta+d-c)^2}{ab}\right))$$

$$= \frac{1}{(ab)^n}\frac{\alpha_0\alpha^n(\alpha+d-c)^n - \beta_0\beta^n(\beta+d-c)^n}{\alpha-\beta}, \tag{2.10}$$

where

$$\alpha_0 = \sum_{l=0}^{3}\frac{a^{\xi(l+1)}}{(ab)^{\left\lfloor\frac{l}{2}\right\rfloor}}(\alpha + d - c)^{\left\lfloor\frac{l+1}{2}\right\rfloor}\alpha^{\left\lfloor\frac{l}{2}\right\rfloor}e_l$$

and

$$\beta_0 = \sum_{l=0}^{3}\frac{a^{\xi(l+1)}}{(ab)^{\left\lfloor\frac{l}{2}\right\rfloor}}(\beta + d - c)^{\left\lfloor\frac{l+1}{2}\right\rfloor}\beta^{\left\lfloor\frac{l}{2}\right\rfloor}e_l.$$

Similarly, we can obtain

$$\Theta_{2n+1} = \frac{1}{(ab)^n}\frac{\alpha_1\alpha^n(\alpha+d-c)^{n+1} - \beta_1\beta^n(\beta+d-c)^{n+1}}{\alpha-\beta}, \tag{2.11}$$

where

$$\alpha_1 = \sum_{l=0}^{3}\frac{a^{\xi(l)}}{(ab)^{\left\lfloor\frac{l+1}{2}\right\rfloor}}(\alpha + d - c)^{\left\lfloor\frac{l}{2}\right\rfloor}\alpha^{\left\lfloor\frac{l+1}{2}\right\rfloor}e_l$$

and

$$\beta_1 = \sum_{l=0}^{3}\frac{a^{\xi(l)}}{(ab)^{\left\lfloor\frac{l+1}{2}\right\rfloor}}(\beta + d - c)^{\left\lfloor\frac{l}{2}\right\rfloor}\beta^{\left\lfloor\frac{l+1}{2}\right\rfloor}e_l.$$

Combining the equations (2.10) and (2.11), we get

$$\Theta_n = \frac{1}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\frac{\alpha_{\xi(n)}\alpha^{\left\lfloor\frac{n}{2}\right\rfloor}(\alpha+d-c)^{n-\left\lfloor\frac{n}{2}\right\rfloor} - \beta_{\xi(n)}\beta^{\left\lfloor\frac{n}{2}\right\rfloor}(\beta+d-c)^{n-\left\lfloor\frac{n}{2}\right\rfloor}}{\alpha-\beta},$$

where

$$\alpha_{\xi(n)} = \sum_{l=0}^{3}\frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\left\lfloor\frac{l+\xi(n)}{2}\right\rfloor}}(\alpha + d - c)^{\left\lfloor\frac{l+1-\xi(n)}{2}\right\rfloor}\alpha^{\left\lfloor\frac{l+\xi(n)}{2}\right\rfloor}e_l$$

and

$$\beta_{\xi(n)} = \sum_{l=0}^{3}\frac{a^{\xi(l+1-\xi(n))}}{(ab)^{\left\lfloor\frac{l+\xi(n)}{2}\right\rfloor}}(\beta + d - c)^{\left\lfloor\frac{l+1-\xi(n)}{2}\right\rfloor}\beta^{\left\lfloor\frac{l+\xi(n)}{2}\right\rfloor}e_l.$$

In the following theorem we derive the Catalan's identity with the help of the Binet formula of Q_n. Furthermore, we give the Cassini's identity which is the special case of the Catalan's identity for $r = 1$.

Theorem 2.4 *(Catalan's identity) For $n, r \in \mathbb{N}_0$ and $r \le n$, we have the identity*

$$\Theta_{2(n+r)+\xi(i)}\Theta_{2(n-r)+\xi(i)} - \Theta_{2n+\xi(i)}^2$$
$$= \frac{(-c)^{\xi(i)}}{(ab)^{2r}(\alpha-\beta)^2}[\alpha_{\xi(i)}\beta_{\xi(i)}((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r)$$
$$+\beta_{\xi(i)}\alpha_{\xi(i)}((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r)],$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.3 and $i \in \{0,1\}$.

Proof. By using the Binet formula of the modified generalized Fibonacci quaternion, for $i = 0$, we get

$$\Theta_{2(n+r)}\Theta_{2(n-r)} - \Theta_{2n}^2$$

$$= \left(\frac{1}{(ab)^{n+r}}\frac{\alpha_0\alpha^{n+r}(\alpha+d-c)^{n+r}-\beta_0\beta^{n+r}(\beta+d-c)^{n+r}}{\alpha-\beta}\right)$$
$$\times \left(\frac{1}{(ab)^{n-r}}\frac{\alpha_0\alpha^{n-r}(\alpha+d-c)^{n-r}-\beta_0\beta^{n-r}(\beta+d-c)^{n-r}}{\alpha-\beta}\right)$$
$$-\left(\frac{1}{(ab)^n}\frac{\alpha_0\alpha^n(\alpha+d-c)^n-\beta_0\beta^n(\beta+d-c)^n}{\alpha-\beta}\right)^2$$
$$= \frac{1}{(ab)^{2n}(\alpha-\beta)^2}(\alpha_0\beta_0(\alpha^n\beta^n(\alpha+d-c)^n(\beta+d-c)^n$$
$$-\alpha^{n+r}\beta^{n-r}(\alpha+d-c)^{n+r}(\beta+d-c)^{n-r})$$
$$+\beta_0\alpha_0(\alpha^n\beta^n(\alpha+d-c)^n(\beta+d-c)^n$$
$$-\alpha^{n-r}\beta^{n+r}(\alpha+d-c)^{n-r}(\beta+d-c)^{n+r}))$$
$$= \frac{1}{(ab)^{2r}(\alpha-\beta)^2}(\alpha_0\beta_0((ab)^{2r}(cd)^n - (ab)^{2r}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r)$$
$$+\beta_0\alpha_0((ab)^{2r}(cd)^n - (ab)^{2r}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r)). \qquad (2.12)$$

Similarly, for $i = 1$, we get

$$\Theta_{2(n+r)+1}\Theta_{2(n-r)+1} - \Theta_{2n+1}^2$$
$$= -\frac{c}{(ab)^{2r}(\alpha-\beta)^2}(\alpha_1\beta_1((ab)^{2r+1}(cd)^n - (ab)^{2r+1}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r)$$
$$+\beta_1\alpha_1((ab)^{2r+1}(cd)^n - (ab)^{2r+1}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r)). \qquad (2.13)$$

By combining the equations (2.12) and (2.13), we obtain

$$\Theta_{2(n+r)+\xi(i)}\Theta_{2(n-r)+\xi(i)} - \Theta_{2n+\xi(i)}^2$$
$$= \frac{(-c)^{\xi(i)}}{(ab)^{2r}(\alpha-\beta)^2}[\alpha_{\xi(i)}\beta_{\xi(i)}((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right)^r)$$
$$+\beta_{\xi(i)}\alpha_{\xi(i)}((ab)^{2r+\xi(i)}(cd)^n - (ab)^{2r+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right)^r)],$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.3 and $i \in \{0,1\}$.

Corollary 2.5 *(Cassini's identity) For $n \in \mathbb{N}_0$, we have the identity*

$$\Theta_{2(n+1)+\xi(i)}\Theta_{2(n-1)+\xi(i)} - \Theta_{2n+\xi(i)}^2$$

$$= \frac{(-c)^{\xi(i)}}{(ab)^2(\alpha-\beta)^2}[\alpha_{\xi(i)}\beta_{\xi(i)}((ab)^{2+\xi(i)}(cd)^n - (ab)^{2+\xi(i)}(cd)^n\left(\frac{\alpha+d}{\beta+d}\right))$$
$$+\beta_{\xi(i)}\alpha_{\xi(i)}((ab)^{2+\xi(i)}(cd)^n - (ab)^{2+\xi(i)}(cd)^n\left(\frac{\beta+d}{\alpha+d}\right))],$$

where $\alpha_{\xi(i)}$ and $\beta_{\xi(i)}$ are defined in Theorem 2.3 and $i \in \{0,1\}$.

The matrix representation of the modified generalized Fibonacci quaternion can be given in the following theorem.

Theorem 2.6 Let $n \geq 1$ be integer. Then

$$\begin{pmatrix} \Theta_{2n+i} & \Theta_{2(n-1)+i} \\ \Theta_{2(n+1)+i} & \Theta_{2n+i} \end{pmatrix} = \begin{pmatrix} \Theta_{2+i} & \Theta_i \\ \Theta_{4+i} & \Theta_{2+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}^{n-1}, \qquad (2.14)$$

where $i \in \{0,1\}$.

Proof. We prove the theorem by induction on n. If $n = 1$ then the result is clear. Now we assume that, for any integer k such as $1 \leq k \leq n$,

$$\begin{pmatrix} \Theta_{2k+i} & \Theta_{2(k-1)+i} \\ \Theta_{2(k+1)+i} & \Theta_{2k+i} \end{pmatrix} = \begin{pmatrix} \Theta_{2+i} & \Theta_i \\ \Theta_{4+i} & \Theta_{2+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}^{k-1}.$$

Then, for $n = k + 1$, by using the identities (2.6) and (2.7) we get

$$\begin{pmatrix} \Theta_{2+i} & \Theta_i \\ \Theta_{4+i} & \Theta_{2+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}^{k}$$

$$= \begin{pmatrix} \Theta_{2+i} & \Theta_i \\ \Theta_{4+i} & \Theta_{2+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}^{k-1}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \Theta_{2k+i} & \Theta_{2(k-1)+i} \\ \Theta_{2(k+1)+i} & \Theta_{2k+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \Theta_{2(k+1)+i} & \Theta_{2k+i} \\ \Theta_{2(k+2)+i} & \Theta_{2(k+1)+i} \end{pmatrix},$$

where $i \in \{0,1\}$. Therefore, the proof is completed.

3. MODIFIED GENERALIZED LUCAS QUATERNIONS

In this section, we define the modified generalized Lucas quaternion ϑ_n. We give the generating function, the Binet formula and some important identities for this quaternion. The theorems and results in this section can be proven similar to the results in Section 2. Hence, we omit the proofs.

Definition 3.1 For $n \in \mathbb{N}_0$, the modified generalized Lucas quaternion ϑ_n is defined by

$$\vartheta_n = \sum_{l=0}^{3} U_{n+l} e_l, \tag{3.1}$$

where U_n is the modified generalized Lucas numbers that is defined in (1.4). Moreover, Eq. (3.1) can be reduced to several quaternions for the special cases of a, b, c and d. For example,
- if we take $c = d = 1$, we obtain the bi-periodic Lucas quaternions in [13],
- if we take $c = d = 1$ and $a = b = k$, we obtain the $k-$Lucas quaternions in [11],
- if we take $c = d = 1$ and $a = b = 1$, we obtain the Lucas quaternions in [10].

Theorem 3.2 The generating function for the modified generalized Lucas quaternion ϑ_n is

$$L(t) = \frac{\vartheta_0 + (\vartheta_1 - a\vartheta_0)t + (b-a)R_1(t) + (d-c)R_2(t)}{1 - at - ct^2}, \tag{3.2}$$

where

$$R_1(t) = e_0 t f(t) + e_1 (f(t) - at) + e_2 \left(\frac{f(t)}{t} - a \right)$$
$$+ e_3 \left(\frac{f(t) - at - (a^2 b + ad + ac + a)t^3}{t^2} \right),$$

$$R_2(t) = e_0 t^2 h(t) + e_1 t \left(h(t) - \left(\frac{d+1}{d} \right) \right) + e_2 \left(h(t) - \left(\frac{d+1}{d} \right) \right)$$
$$+ e_3 \left(\frac{h(t) - \left(\frac{d+1}{d} \right) - (ab + d + 1)t^2}{t} \right),$$

$$f(t) = \frac{at + at^3}{1 - (ab + d + c)t^2 + cdt^4}$$

and

$$h(t) = \frac{\left(\frac{d+1}{d} \right) + (ab + d + 1)t^2 - (ab + d + c)\left(\frac{d+1}{d} \right)t^2}{1 - (ab + d + c)t^2 + cdt^4}.$$

Theorem 3.3 For $n \in \mathbb{N}_0$, the Binet formula for the modified generalized Lucas quaternion is

$$\vartheta_n = \frac{1}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \left(\frac{\alpha^*_{\xi(n)}(\alpha + d + 1)\alpha^{\lfloor \frac{n-1}{2} \rfloor}(\alpha + d - c)^{\lfloor \frac{n}{2} \rfloor}}{\alpha - \beta} - \frac{\beta^*_{\xi(n)}(\beta + d + 1)\beta^{\lfloor \frac{n-1}{2} \rfloor}(\beta + d - c)^{\lfloor \frac{n}{2} \rfloor}}{\alpha - \beta} \right), \tag{3.3}$$

where

$$\alpha^*_{\xi(n)} = \sum_{l=0}^{3} \frac{a^{\xi(l+\xi(n))}}{(ab)^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor}} (\alpha + d - c)^{\lfloor \frac{l+\xi(n)}{2} \rfloor} \alpha^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor} e_l$$

and

$$\beta^*_{\xi(n)} = \sum_{l=0}^{3} \frac{a^{\xi(l+\xi(n))}}{(ab)^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor}} (\beta + d - c)^{\lfloor \frac{l+\xi(n)}{2} \rfloor} \beta^{\lfloor \frac{l+1-\xi(n)}{2} \rfloor} e_l.$$

In the following theorem we derive the Catalan's identity with the help of the Binet formula of U_n. Furthermore, we give the Cassini's identity which is the special case of the Catalan's identity for $r = 1$.

Theorem 3.4 (Catalan's identity) For $n, r \in \mathbb{N}_0$ and $r \leq n$, we have the identity

$$\vartheta_{2(n+r)+\xi(i)}\vartheta_{2(n-r)+\xi(i)} - \vartheta_{2n+\xi(i)}^2$$

$$= \frac{(-c)^{1-\xi(i)}(\alpha+d+1)(\beta+d+1)}{(ab)^{2r}(\alpha-\beta)^2}$$
$$\times [\alpha_{\xi(i)}^*\beta_{\xi(i)}^*((ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)}$$
$$-(ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)^r)$$
$$+\beta_{\xi(i)}^*\alpha_{\xi(i)}^*((ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)}$$
$$-(ab)^{2r+1-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right)^r)],$$

where $\alpha_{\xi(i)}^*$ and $\beta_{\xi(i)}^*$ are defined in Theorem 3.3 and $i \in \{0,1\}$.

Corollary 3.5 (Cassini's identity) For $n \in \mathbb{N}_0$, we have the identity

$$\vartheta_{2(n+1)+\xi(i)}\vartheta_{2(n-1)+\xi(i)} - \vartheta_{2n+\xi(i)}^2$$

$$= \frac{(-c)^{1-\xi(i)}(\alpha+d+1)(\beta+d+1)}{(ab)^2(\alpha-\beta)^2} \times [\alpha_{\xi(i)}^*\beta_{\xi(i)}^*((ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)}$$
$$-(ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\alpha+d}{\beta+d}\right)) + \beta_{\xi(i)}^*\alpha_{\xi(i)}^*((ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)}$$
$$-(ab)^{3-\xi(i)}(cd)^{n-1+\xi(i)}\left(\frac{\beta+d}{\alpha+d}\right))],$$

where $\alpha_{\xi(i)}^*$ and $\beta_{\xi(i)}^*$ are defined in Theorem 3.3 and $i \in \{0,1\}$.

Theorem 3.6 Let $n \geq 1$ be integer. Then the modified generalized Lucas quaternion satisfies the relation

$$\vartheta_n = \Theta_{n-1} + \Theta_{n+1}. \tag{3.4}$$

Proof. By considering the identity $U_n = Q_{n-1} + Q_{n+1}$, which is given by the Theorem 20 in [7], we can easily obtain the desired result.

Theorem 3.7 Let $n \geq 1$ be integer. Then

$$\begin{pmatrix} \vartheta_{2n+i} & \vartheta_{2(n-1)+i} \\ \vartheta_{2(n+1)+i} & \vartheta_{2n+i} \end{pmatrix} = \begin{pmatrix} \vartheta_{2+i} & \vartheta_i \\ \vartheta_{4+i} & \vartheta_{2+i} \end{pmatrix}\begin{pmatrix} ab+c+d & 1 \\ -cd & 0 \end{pmatrix}^{n-1}, \tag{3.5}$$

where $i \in \{0,1\}$.

Proof. By considering the identities in Lemma 2 in [7], the proof can be made similar to the Theorem 2.6.

4. CONCLUSION

In this paper, we define the modified generalized Fibonacci and modified generalized Lucas quaternions. These quaternions are the generalization of several studies in the literature such as [10-15]. Moreover, we give the Catalan's identity, Cassini's identity and matrix representations in more general form. Since our study both generalization of several studies in the literature and includes some new results, it contributes to the literature by providing essential information on the generalization of the quaternions.

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