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On the extensions of the almost convergence idea and core theorems

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Abstract

The sequence spaces rf and rf_0, more general and comprehensive than the almost convergent sequence spaces f and f_0, were introduced by Zararsız and Şengönül in [Z. Zararsız, M. Şengönül, Doctoral Thesis, Nevşehir, (2015)]. The purpose of the present paper is to study the sequence spaces brf and brf_0, that is, the sets of all sequences such that their $\mathscr{B}(r,s)$ transforms are in rf and rf_0 respectively. Furthermore, we determine the β- and γ- duals of brf, we show that there exists a linear isomorphic mapping between the spaces rf and brf, and between rf_0 and brf_0 respectively, and provide some matrix characterizations of these spaces. Finally, we introduce the $B_{R\mathscr{B}} - core$ of a complex valued sequence and prove some theorems related to this new type of core. ©2016 All rights reserved.

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1. Preliminaries, background and notations

There exist various ways of constructing new spaces from a given sequence space. One of them concerns the convergence fields of any infinite matrix. Using this method, many sequence spaces have been constructed (see, e.g., [1]-[4], [6]-[10], [13],[15], [17]-[24], [28]-[34], [37]). Another way to obtain a new space is by using standard techniques. Spaces constructed by means of these techniques are called base spaces. For example, ℓ_∞, c, c_0, ℓ_p, f and f_0, i.e. the spaces of bounded, convergent, null, absolutely p summable $(1 \leq p < \infty)$, almost convergent and almost null convergent sequences of complex numbers respectively, are considered as base spaces. Zararsız and Şengönül ([39]) have constructed a new base sequence space called the space of rf - convergent sequences and investigated some of its important properties.

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In the present paper, by using matrix domains, we define the spaces of brf - convergent and null brf - convergent sequences. In order to explain this concept, we provide some necessary notations and definitions. For brevity in notation, through all the text we will write $\sum_n$, $\sup_n$, $\limsup_n$ and $\lim_n$ instead of $\sum_{n=0}^{\infty}$, $\sup_{n\in\mathbb{N}}$, $\limsup_{n\to\infty}$ and $\lim_{n\to\infty}$, where $\mathbb{N}=\{0,1,2,...\}$. Furthermore, we denote by $\mathbb{R}$ and $\mathbb{C}$ the set of real and complex numbers, respectively. The set of all real and complex valued sequences, which is denoted in the following by w, is a linear space with the addition and scalar multiplication of sequences. Each linear subspace of w is called a sequence space. We will use the notations bs and cs for the spaces of all bounded and convergent series, respectively.

Let λ and μ be two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k\in\mathbb{N}$. Then, we say that A defines a matrix mapping from λ to μ, denoted $A\in(\lambda:\mu)$, if for every sequence $x=(x_k)$ in λ, the sequence $Ax=((Ax)_n)$, called the A-transform of x, is in μ. The domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A=\{x=(x_k)\in w : Ax\in\lambda\}, \tag{1.1}$$

which is a sequence space. If we take $\lambda=c$, then c_A is called the convergence domain of A. We write the limit of Ax as $A-\lim_n x_n=\lim_n \sum_{k=0}^{\infty} a_{nk}x_k$, and A is called regular if $\lim Ax=\lim x$ for every $x\in c$. $A=(a_{nk})$ is said to be a triangle matrix if $a_{nk}=0$ for $k>n$ and $a_{nn}\neq 0$ for all $n\in\mathbb{N}$. If A is a triangle matrix, then one can easily see that the sequence spaces λ_A and λ are linearly isomorphic, i.e. $\lambda_A\cong\lambda$. A sequence space λ with a linear topology is called a K- space provided each of the maps $p_i:\lambda\to\mathbb{C}$ defined by $p_i(x)=x_i$ is continuous for all $i\in\mathbb{N}$. If λ is a complete linear metric space then it is called an FK- space. Any FK-space whose topology is normable is called a BK- space [12]. If a normed sequence space λ has a sequence (b_n) with the property that for every $x\in\lambda$ there exists a unique sequence of scalars (a_n) such that

$$\lim_{n\to\infty}\|x-(a_0b_0+a_1b_1+\cdots+a_nb_n)\|=0$$

then (b_n) is called a Schauder basis for λ.

Next we recall some particular some triangle and regular matrices which we will need in the following. The Cesàro matrix of order one is the lower triangular matrix $C=(c_{nk})$ defined by

$$c_{nk}=\begin{cases}\frac{1}{n+1}, & 0\leq k\leq n,\\ 0, & k>n\end{cases}$$

for all $n,k\in\mathbb{N}$.

The Riesz matrix $R=(r_{nk})$ is defined by

$$r_{nk}=\begin{cases}\frac{r_k}{R_n}, & 0\leq k\leq n,\\ 0, & k>n\end{cases}$$

for all $n,k\in\mathbb{N}$, where (r_k) is a real sequence with $r_0>0$, $r_k\geq 0$ and $R_n=\sum_{k=0}^{n} r_k$. It is known that the Riesz matrix R is regular if and only if $R_n\to\infty$ as $n\to\infty$ [32].

Now, let r,s be non-zero real numbers and define the generalized difference matrix $\mathscr{B}(r,s)=(b_{nk}(r,s))$ by

$$b_{nk}(r,s)=\begin{cases}r, & k=n,\\ s, & k=n-1,\\ 0, & 0\leq k<n-1 \text{ or } k>n\end{cases} \qquad (k,n\in\mathbb{N}). \tag{1.2}$$

One can determine by a straightforward calculation that the inverse $\mathscr{B}^{-1}(r,s)=(\hat{b}_{nk}(r,s))$ of the generalized difference matrix is given by

$$\hat{b}_{nk}(r,s)=\begin{cases}\frac{1}{r}\left(-\frac{s}{r}\right)^{n-k}, & 0\leq k\leq n,\\ 0, & k>n,\end{cases} \qquad (k,n\in\mathbb{N}).$$

The aim of the present paper is to introduce the space *brf* and to investigate some of its properties.

The outline of the paper is as follows. In Section 2, we present the concepts of almost convergence and space of almost convergent and almost null sequences. Section 3 is devoted to the spaces *brf* and brf_0. Additionally, we define the T- convergent and null T- convergent sequence spaces and give some properties of the spaces *brf* and brf_0. In Section 4, we determine the β- and γ-duals of *brf*. In Section 5, we characterize some matrix mappings from *brf* to any given sequence space μ and vice versa via the dual summability methods. In the final section, we introduce the $B_{R\mathscr{B}}-core$ of a complex valued sequence and obtain some theorems related to this new type of core.

2. The space of almost convergent sequences

We recall some required definitions and results from the paper [26] of Lorentz. The shift operator S on ℓ_∞ is defined by $(Sx)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is a non-negative linear functional on ℓ_∞ satisfying $L(Sx) = L(x)$ and $L(e) = 1$ where $e = (1, 1, 1, ...)$. A bounded sequence is called almost convergent to the generalized limit a if all Banach limits of the sequence x are equal to a. This is denoted by $f - \lim x = a$. It was shown in [26] that $f - \lim x = a$ if and only if $\lim\limits_p \frac{(x_n+x_{n+1}+...+x_{n+p-1})}{p} = a$, uniformly in n. By f and f_0 we denote the space of all almost convergent and almost null sequences respectively, i.e.

$$f = \left\{ x = (x_k) \in \ell_\infty : \exists a \in \mathbb{C} \ni \lim_m \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = a, \text{ uniformly in } n \right\},$$

and

$$f_0 = \left\{ x = (x_k) \in \ell_\infty : \lim_m \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = 0, \text{ uniformly in } n \right\}.$$

In the aforementioned paper, Lorentz obtained the necessary and sufficient conditions for an infinite matrix to contain f in its convergence domain. These conditions are standard Silverman Toeplitz conditions for regularity, plus the requirement that

$$\lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{nk+1}| = 0. \tag{2.1}$$

A matrix U is called a *generalized Cesàro matrix* if it is obtained from the matrix C by shifting rows. Let $\theta : \mathbb{N} \to \mathbb{N}$. Then $U = (u_{nk})$ is defined by

$$u_{nk} = \begin{cases} \frac{1}{n+1}, & \theta(n) \le k \le \theta(n) + n, \\ 0, & \text{otherwise} \end{cases}$$

for all $n,\ k \in \mathbb{N}$.

Let us denote by G the set of all such matrices obtained by using all possible functions θ. This set can be used to characterize the space f, as shown by the next lemma given by Butkovic, Kraljevic and Sarapa in [16].

Lemma 2.1 ([16])**.** *The set f of all almost convergent sequences is equal to the set $\cap_{U\in G} c_U$.*

3. The sequence spaces *brf* and brf_0

In this section, we introduce the sequence spaces *brf* and brf_0 derived by the domain of the matrix $\mathscr{B}(r, s)$. Also, we determine an isomorphism between the spaces *brf* and *rf*, and between brf_0 and rf_0.

As seen above, almost convergence can be defined as the intersection of convergence field of a matrix that is obtained by displacement of the lines of the first-order Cesàro matrix. Let $v \in \mathbb{N}$ and $x = (x_k) \in \ell_\infty$. Let us define the matrix $S^v = (s^v_{nk})$ as follows:

$$s^v_{nk} = \begin{cases} 1, & n + v = k, \\ 0, & \text{otherwise.} \end{cases}$$

Using S, we construct $(S^v x) = (S^0 x, S^1 x, S^2 x, ..., S^v x, ...)$, the sequence of shifted transforms of x. Thus, almost convergence has the same meaning as the convergence of first-order Cesàro average of the shifted transform sequence $(S^v x) = (S^0 x, S^1 x, S^2 x, ..., S^v x, ...)$ to a fixed sequence for each v. We can now generalize f and f_0 by considering the following sequence spaces called the sets of all T-convergent and null T-convergent sequences, respectively:

$$f_T = \left\{ x \in \ell_\infty : \lim_k [T(S^v x)]_k = \ell \in \mathbb{C}, v = 0, 1, 2, ... \right\},$$

$$f_{T_0} = \left\{ x \in \ell_\infty : \lim_k [T(S^v x)]_k = 0, v = 0, 1, 2, ... \right\}.$$

By considering $T = R = (r_{nk})$ in the definition of the sets f_T and f_{T_0}, we obtain the concepts of rf- convergent and null rf- convergent sequence spaces introduced by Zararsız and Şengönül in [39], namely

$$rf = \left\{ x = (x_k) \in \ell_\infty : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} = a, \text{ uniformly in } n \right\}, \tag{3.1}$$

and

$$rf_0 = \left\{ x = (x_k) \in \ell_\infty : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} = 0, \text{ uniformly in } n \right\}. \tag{3.2}$$

Since the spaces rf and rf_0 are not obtained by the convergence field of an infinite matrix, these spaces can be seen as base spaces. In addition to rf and rf_0, we define two new types of spaces of convergent sequences, brf and brf_0, as the sets of all sequences such that their $\mathscr{B}(r,s)$- transforms are in the spaces rf and rf_0 respectively, that is,

$$brf = \left\{ x = (x_k) \in w : \lim_m \sum_{k=0}^{m} \frac{r_k}{R_m} [s x_{k+n-1} + r x_{k+n}] = a \in \mathbb{C}, \text{ uniformly in } n \right\},$$

and

$$brf_0 = \left\{ x = (x_k) \in w : \lim_m \sum_{k=0}^{m} \frac{r_k}{R_m} [s x_{k+n-1} + r x_{k+n}] = 0, \text{ uniformly in } n \right\}.$$

Using the notation from (1.1), we can redefine the spaces brf and brf_0 as

$$brf = (rf)_{\mathscr{B}(r,s)} \text{ and } brf_0 = (rf_0)_{\mathscr{B}(r,s)}.$$

It is clear that brf and brf_0 are not base spaces. Now, let us define the sequence $y = (y_k)$ as the $\mathscr{B}(r,s)$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = s x_{k-1} + r x_k, \quad (k \in \mathbb{N}). \tag{3.3}$$

We should emphasize here that the sequence spaces rf and rf_0 can be reduced to the classical almost convergent sequence spaces of real numbers f and f_0 respectively if we set $r_k = 1$ for all $k \in \mathbb{N}$. Thus, the properties and results related to the sequence spaces rf and rf_0 are more general than the corresponding implications regarding f and f_0.

Lorentz ([26]) proved that if the regular matrix method A has the property (2.1), then f and rf are equivalent. However, if $\lim_n R_n$ is not equal to ∞, then the Riesz matrix R is not a Toeplitz matrix. Therefore, in general, the spaces rf and rf_0 are different from the spaces f and f_0.

Lemma 3.1 ([39])**.** *The sets rf and rf_0 are Banach spaces with the norm*

$$\|x\|_{rf} = \|x\|_{rf_0} = \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} \right|, \text{ uniformly in } n. \tag{3.4}$$

Corollary 3.2 ([39]). *The space rf has no Schauder basis.*

Theorem 3.3. *The sets brf and brf_0 are linear spaces with the co-ordinatewise addition and scalar multiplication. Additionally, they are BK- spaces with the norm defined by*

$$\|x\|_{brf_0} = \|x\|_{brf} = \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k [s x_{k+n-1} + r x_{k+n}] \right|, \text{ uniformly in } n. \tag{3.5}$$

Proof. The first part of the theorem is clear so we will only prove the second claim. Since (3.3) holds, *rf* and rf_0 are Banach spaces (see Lemma 3.1) and the matrix $\mathscr{B}(r,s)$ is normal, Theorem 4.3.3 of Wilansky [38] ensures that *brf* and brf_0 are BK- spaces. □

Theorem 3.4. *The sequence spaces rf and rf_0 are isometrically isomorphic to the spaces brf and brf_0 respectively, that is, $rf \cong brf$ and $rf_0 \cong brf_0$.*

Proof. To avoid the repetition of similar statements, we give the proof only for the spaces *rf* and *brf*. In order to prove the fact that $rf \cong brf$, we should show the existence of a linear bijection between these spaces. Consider the transformation T defined using the notation of (3.3), from *brf* to *rf*, by $x \mapsto y = Tx$. The linearity of T is clear. Furthermore, it is trivial that x equals $\theta = (0, 0, ...)$ whenever $Tx = \theta$ and hence T is injective.

Let $y = (y_k) \in rf$ and define the sequence $x = (x_k)$ by

$$x = (x_k) = (\{\mathscr{B}^{-1}(r,s)y\})_k = \sum_{j=0}^{k} \frac{1}{r} \left(-\frac{s}{r}\right)^j y_{k-j}$$

for all $k \in \mathbb{N}$. Then it is easy to see that

$$s x_{k-1} + r x_k = \sum_{j=0}^{k-1} \frac{s}{r} \left(-\frac{s}{r}\right)^j y_{k-j-1} + \sum_{j=0}^{k} \left(-\frac{s}{r}\right)^j y_{k-j} = y_k$$

for all $k \in \mathbb{N}$, which means that

$$\lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k (s x_{k+n-1} + r x_{k+n}) = \lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k y_{k+n} = brf - \lim y_k, \text{ uniformly in } n.$$

Thus, $x = (x_k) \in brf$. Consequently, it is clear that T is surjective. Now, we show that T is norm preserving, which will imply that the spaces *brf* and *rf* are isometrically isomorphic as desired:

$$\begin{aligned}
\|x\|_{brf} &= \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k [s x_{k+n-1} + r x_{k+n}] \right| \\
&= \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k \left[\frac{s}{r} \sum_{j=0}^{k-1} \left(-\frac{s}{r}\right)^j y_{k+n-j-1} + \sum_{j=0}^{k} \left(-\frac{s}{r}\right)^j y_{k+n-j} \right] \right| \\
&= \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k y_{k+n} \right| = \|y\|_{rf}.
\end{aligned}$$

This concludes the proof. □

It is known that a set $\lambda \subset w$ is said to be convex if $x, y \in \lambda$ implies $M = \{z \in w : z = tx + (1-t)y, 0 \leq t \leq 1\} \subset \lambda$. In the case of the spaces *rf*, rf_0, *brf* and brf_0 we have

Theorem 3.5. *The sets rf, rf_0, brf and brf_0 are convex spaces.*

Proof. The proof of the theorem is clear from the definition of convexity. □

The matrix domain λ_A of a sequence space λ has a basis if and only if λ has a basis and $A = (a_{nk})$ is a triangle matrix. Thus, by Corollary 3.2 we obtain

Corollary 3.6. *The space brf has no Schauder basis.*

4. Duals

In this section, by using the techniques in [5], we state and prove the theorems determining the β- and γ-duals of the spaces brf_0 and brf.

For the sequence spaces λ and μ, define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x = (x_k) \in \lambda\}. \tag{4.1}$$

If we take $\mu = \ell_1$ then the set $S(\lambda, \ell_1)$ is called the α-dual of λ, and similarly the sets $S(\lambda, cs)$, $S(\lambda, bs)$ are called β- and γ-dual of λ and denoted by λ^β and λ^γ, respectively.

The following results will be used in the computation of the β- dual of the sets brf and brf_0.

Lemma 4.1 ([39])**.** *Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (rf : \ell_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{4.2}$$

Proposition 4.2 ([39])**.** *Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (rf : c)$ if and only if*

$$\lim_n \sum_k a_{nk} = a, \tag{4.3}$$

$$\lim_n a_{nk} = a_k \quad (k \in \mathbb{N}), \tag{4.4}$$

and

$$\lim_n \sum_k |\triangle(a_{nk} - a_k)| = 0. \tag{4.5}$$

Lemma 4.3 ([39])**.** *Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (\ell_\infty : rf)$ if and only if relations* (4.2),

$$rf - \lim_n a_{nk} = a_k, \forall k \in \mathbb{N}, \tag{4.6}$$

and

$$\lim_m \sum_k \left| \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k} - a_k \right| = 0, \textit{ uniformly in } n \tag{4.7}$$

hold.

Lemma 4.4 ([39])**.** *Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (c : rf)$ if and only if the relations*

$$\sup_m \sum_k \left| \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{ik} \right| < \infty \quad (k, m \in \mathbb{N}), \tag{4.8}$$

$$\lim_m \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k} = a_k, \textit{ uniformly in } n, \ (a_k \in \mathbb{C}) \tag{4.9}$$

and

$$\lim_m \frac{1}{R_m} \sum_k \sum_{i=0}^{m} r_i a_{n+i,k} = a, \ \textit{ uniformly in } n \tag{4.10}$$

hold.

Lemma 4.5 ([5])**.** *Let $D = (d_{nk})$ be defined via the sequence $a = (a_k) \in w$ and the inverse $V = (v_{nk})$ of the triangle matrix $U = (u_{nk})$, by*

$$d_{nk} = \begin{cases} \sum\limits_{j=k}^{n} a_j v_{jk}, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. Then

$$\{\lambda_U\}^{\gamma} = \{a = (a_k) \in w : D \in (\lambda : \ell_\infty)\},$$

and

$$\{\lambda_U\}^{\beta} = \{a = (a_k) \in w : D \in (\lambda : c)\}.$$

Theorem 4.6. *The γ-dual of the space brf is the set $d_1(r, s)$ defined by*

$$d_1(r,s) = \left\{ a = (a_k) \in w : \sup_n \sum_{k=0}^{n} \left| \frac{1}{r} \sum_{j=k}^{n} \left(-\frac{s}{r}\right)^{j-k} a_j \right| < \infty \right\}.$$

Proof. The proof is clear, so we omit it. □

Let us define the sets d_i for $i = 2, 3, 4$ as follows:

$$d_2(r,s) = \left\{ a = (a_k) \in w : \lim_n \sum_{j=k}^{n} \frac{1}{r} \left(-\frac{s}{r}\right)^{j-k} a_j \text{ exists} \right\},$$

$$d_3(r,s) = \left\{ a = (a_k) \in w : \lim_n \left| \Delta \left(\sum_{j=k}^{n} \frac{1}{r} \left(-\frac{s}{r}\right)^{j-k} a_j - a_k \right) \right| = 0 \right\},$$

$$d_4(r,s) = \left\{ a = (a_k) \in w : \lim_n \sum_{k=0}^{n} \left[\frac{1 - \left(-\frac{s}{r}\right)^{k+1}}{1 + \frac{s}{r}} a_k \right] \text{ exists} \right\}.$$

Theorem 4.7. *The β-dual of the space brf is the set $\mathscr{D} = \bigcap_{i=1}^{4} d_i(r,s)$.*

Proof. Define the matrix $V = (v_{nk})$ via the sequence $u = (u_k) \in w$ by

$$v_{nk} = \begin{cases} \sum\limits_{j=k}^{n} \frac{1}{r} \left(-\frac{s}{r}\right)^{j-k} u_j, & 0 \le k \le n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Taking into account that $x_k = \sum\limits_{j=k}^{n} \frac{1}{r} \left(-\frac{s}{r}\right)^{j-k} y_j$, we find that

$$\sum_{k=0}^{n} u_k x_k = \sum_{k=0}^{n} \sum_{j=k}^{n} \frac{1}{r} \left(-\frac{s}{r}\right)^{j-k} u_j y_k = (Vy)_n \quad (n \in \mathbb{N}). \tag{4.11}$$

From (4.11), we see that $ux = (u_k x_k) \in cs$ whenever $x = (x_k) \in brf$ if and only if $Vy \in c$ whenever $y = (y_k) \in rf$. Then we derive by Proposition 4.2 that $brf^{\beta} = \mathscr{D}$. □

5. Some matrix mappings related to the space *brf*

In the present section, we characterize the matrix mappings from *brf* into any given sequence space μ and vice versa via the concept of dual summability methods of the new type.

Dual summability methods have been used by many authors, such as Başar [11], Başar and Çolak [14], Kuttner [25], Lorentz and Zeller [27]. We provide a brief overview of these methods, following Başar [11].

Let us suppose that the sequences $u = (u_k)$ and $v = (v_k)$ are connected via relation (3.3), and let the A-transform of the sequence $u = (u_k)$ be $z = (z_k)$ and the B-transform of the sequence $v = (v_k)$ be $t = (t_k)$, i.e.

$$z_k = (Au)_k = \sum_k a_{nk}u_k, \quad (k \in \mathbb{N}) \tag{5.1}$$

and

$$t_k = (Bv)_k = \sum_k b_{nk}v_k, \quad (k \in \mathbb{N}). \tag{5.2}$$

It is clear here that the method B is applied to the $\mathscr{B}(r,s)$- transform of the sequence $u = (u_k)$ while the method A is directly applied to the terms of the sequence $u = (u_k)$. So, the methods A and B are essentially different (see [11]).

Let us assume that the matrix product $B\mathscr{B}(r,s)$ exists. This is a much weaker assumption than the conditions on the matrix B belonging to any matrix class, in general. If z_k becomes t_k (or t_k becomes z_k), under the application of the formal summation by parts, then the methods A and B as in (5.1) and (5.2) are called generalized difference dual type matrices. This leads us to fact that $B\mathscr{B}(r,s)$ exists and is equal to A and $(B\mathscr{B}(r,s))u = B(\mathscr{B}(r,s)u)$ formally holds. This statement is equivalent to the relation

$$b_{nk} = \sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} a_{nj} \text{ or } a_{nk} = sb_{n,k-1} + rb_{nk} \tag{5.3}$$

for all $n, k \in \mathbb{N}$.

Now we can give the following theorem concerning generalized difference dual matrices:

Theorem 5.1. *Let $A = (a_{nk})$ and $B = (b_{nk})$ be dual matrices of the new type and μ be any given sequence space. Then, $A \in (brf : \mu)$ if and only if $(a_{nk})_{k\in\mathbb{N}} \in brf^{\beta}$ for all $n \in \mathbb{N}$ and $B \in (rf : \mu)$.*

Proof. Suppose that $A = (a_{nk})$ and $B = (b_{nk})$ are generalized difference dual matrices, that is to say that (5.3) holds, and let μ be any given sequence space. Let us keep in mind that the spaces *brf* and *rf* are isomorphic.

Let $A \in (brf : \mu)$ and take any $y = (y_k) \in rf$. Then $B\mathscr{B}(r,s)$ exists and $(a_{nk})_{k\in\mathbb{N}} \in \mathscr{D}$, which yields that $(b_{nk})_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence By exists, and since

$$\sum_k b_{nk}y_k = \sum_k a_{nk}x_k \tag{5.4}$$

for all $n \in \mathbb{N}$, we have $By = Ax$, whence $B \in (rf : \mu)$. Conversely, let $(a_{nk})_{k\in\mathbb{N}} \in brf^{\beta}$ for each $n \in \mathbb{N}$ and $B \in (rf : \mu)$, and take any $x = (x_k) \in brf$. Then it is clear that Ax exists. From here, given that

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m}\left[\sum_{j=k}^{m}\frac{1}{r}\left(-\frac{s}{r}\right)^{j-k}a_{nj}\right]y_k = \sum_{k=0}^{m} b_{nk}y_k$$

for all $n \in \mathbb{N}$, we can infer that $Ax = By$, and this shows that $A \in (brf : \mu)$. □

Theorem 5.2. *Suppose that the elements of the infinite matrices $D = (d_{nk})$ and $E = (e_{nk})$ satisfy the relation*

$$e_{nk} = sd_{n-1,k} + rd_{nk} \tag{5.5}$$

for all $n, k \in \mathbb{N}$, and let μ be any given sequence space. Then $D \in (\mu : brf)$ if and if only $E \in (\mu : rf)$.

Proof. Let us suppose that $x = (x_k) \in \mu$ and consider the following equality with (5.5):

$$\begin{aligned}\{\mathscr{B}(r,s)(Dx)\}_n &= s(Dx)_{n-1} + r(Dx)_n \\ &= s\sum_k d_{n-1,k}x_k + r\sum_k d_{nk}x_k \\ &= \sum_k (sd_{n-1,k} + rd_{nk})x_k \\ &= (Ex)_n.\end{aligned}$$

This yields that $Dx \in brf$ if and only if $Ex \in rf$. This completes the proof. □

The results we present next can be obtained from Proposition 4.2, Lemma 4.3 and Theorems 5.1 and 5.2. Consider the following statements:

$$\sup_n \sum_k \left| \sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} a_{nj} \right| < \infty, \tag{5.6}$$

$$\lim_n \sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} a_{nj} = a_k, \tag{5.7}$$

$$\lim_n \sum_k \sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} a_{nj} = a, \text{ for each fixed } k \in \mathbb{N}, \tag{5.8}$$

$$\lim_n \sum_k \left| \Delta\left(\sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} a_{nj} - a_k \right) \right| = 0, \text{ for each fixed } k \in \mathbb{N}, \tag{5.9}$$

$$rf - \lim_n \sum_{j=k}^{n} \frac{1}{r}\left(-\frac{s}{r}\right)^{j-k} a_{nj} = a_k \text{ exists for each fixed } k \in \mathbb{N}, \tag{5.10}$$

$$\lim_m \sum_k \left| \frac{1}{R_m} \sum_{j=0}^{m} r_k a_{n+j,k} - a_k \right| = 0, \text{ uniformly in } n, \tag{5.11}$$

$$\sup_n \sum_k |sa_{n-1,k} + ra_{nk}| < \infty, \tag{5.12}$$

$$rf - \lim_n (sa_{n-1,k} + ra_{nk}) = \alpha_k \text{ exists for each } k \in \mathbb{N}, \tag{5.13}$$

$$rf - \lim_n \sum_k (sa_{n-1,k} + ra_{nk}) = \alpha. \tag{5.14}$$

Proposition 5.3. *The following statements hold:*

(i) $A = (a_{nk}) \in (brf : \ell_\infty)$ *if and only if* $(a_{nk})_{k\in\mathbb{N}} \in brf^\beta$ *for all* $n \in \mathbb{N}$ *and* (5.6) *holds.*
(ii) $A = (a_{nk}) \in (brf : c)$ *if and only if* $(a_{nk})_{k\in\mathbb{N}} \in brf^\beta$ *for all* $n \in \mathbb{N}$ *and* (5.6), (5.7), (5.8) *and* (5.9) *hold.*
(iii) $A = (a_{nk}) \in (\ell_\infty : brf)$ *if and only if* (5.11), (5.12) *and* (5.13) *hold.*
(iv) $A = (a_{nk}) \in (c : brf)$ *if and only if* (5.12), (5.13) *and* (5.14) *hold.*

6. *brf* - type core theorems

In this section, we give some core theorems related to the *rf* - and *brf* - cores. Let $x = (x_k)$ be a sequence in $\mathbb{C}$ and $\mathfrak{R}_k$ be the least convex closed region of the complex plane containing $x_k, x_{k+1}, x_{k+2}, \ldots$. The Knopp Core (or $\mathcal{K} - core$) of x is defined by the intersection of all $\mathfrak{R}_k$ $(k = 1, 2, \ldots)$ (see [21]). In [35], it is shown that

$$\mathcal{K} - core(x) = \bigcap_{z \in \mathbb{C}} B_x(z)$$

for any bounded sequence x, where $B_x(z) = \{w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z|\}$.

Let E be a subset of $\mathbb{N}$. The natural density δ of E is defined by $\delta(E) = \lim\limits_n \frac{1}{n}|\{k \leq n : k \in E\}|$ where $|\{k \leq n : k \in E\}|$ denotes the number of elements of E not exceeding n. A sequence $x = (x_k)$ is said to be statistically convergent to a number ℓ if $\delta(\{k : |x_k - \ell| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$. In this case, we write $st - \lim x = \ell$ ([36]). By st and st_0, we denote the space of all statistically convergent and statistically null sequences, respectively.

In [23], Fridy and Orhan introduced the notion of the statistical core(or $st - core$) of a complex valued sequence and showed that, for a statistically bounded sequence x,

$$st - core(x) = \bigcap_{z \in \mathbb{C}} C_x(z),$$

where $C_x(z) = \{w \in \mathbb{C} : |w - z| \leq st - \limsup\limits_k |x_k - z|\}$.

Here, we consider sequences with complex entries and by $\ell_\infty(\mathbb{C})$ we denote the space of all bounded complex valued sequences. Following Knopp, a core theorem is characterized by a class of matrices for which the core of the transformed sequence is included in the core of original sequence.

Now, we introduce the $B_{R\mathscr{B}} - core$ of a complex valued sequence and characterize the class of matrices to yield $B_{R\mathscr{B}} - core(Ax) \subseteq \mathcal{K} - core(x)$, $\mathcal{K} - core(Ax) \subseteq B_{R\mathscr{B}} - core(x)$, $B_{R\mathscr{B}} - core(Ax) \subseteq B_{R\mathscr{B}} - core(x)$ and $B_{R\mathscr{B}} - core(Ax) \subseteq st - core(x)$ for all $x \in \ell_\infty(\mathbb{C})$.

Let us set

$$t_{mn}(x) = \frac{1}{R_m} \sum_{i=0}^{m} r_i [s x_{i+n-1} + r x_{i+n}].$$

Then, we can define the $B_{R\mathscr{B}} - core$ of a complex sequence as follows:

Definition 6.1. Let H_n be the least closed convex hull containing $t_{mn}(x), t_{m+1,n}(x), t_{m+2,n}(x), \ldots$. Then

$$B_{R\mathscr{B}} - core(x) = \bigcap_{n=1}^{\infty} H_n.$$

Note that we define the $B_{R\mathscr{B}} - core$ of x by the $\mathcal{K} - core$ of the sequence $(t_{mn}(x))$. Hence, we can obtain the following theorem which is analogue of a result in [35]:

Theorem 6.2. *For any $z \in \mathbb{C}$, let*

$$G_x(z) = \left\{ w \in \mathbb{C} : |w - z| \leq \limsup_m \sup_n |t_{mn}(x) - z| \right\}.$$

Then, for any $x \in \ell_\infty$,

$$B_{R\mathscr{B}} - core(x) = \bigcap_{z \in \mathbb{C}} G_x(z).$$

Next, we characterize the classes $A \in (c{:}brf)_{reg}$ and $(st \bigcap \ell_\infty : brf)_{reg}$. For brevity, through all the text we write $\tilde{a}(m,n,k)$ in place of

$$\frac{1}{R_m}\sum_{i=0}^{m} r_i[sa_{i+n-1,k} + ra_{i+n,k}]$$

for all $m,n,k \in \mathbb{N}$.

Lemma 6.3. *$A \in (c : brf)_{reg}$ if and only if (5.6) and (5.10) hold with $a_k = 0$ for all $k \in \mathbb{N}$ and*

$$\lim_m \sum_k \tilde{a}(m,n,k) = 1, \text{ uniformly in } n. \tag{6.1}$$

Lemma 6.4. *$A \in (st \bigcap \ell_\infty : brf)_{reg}$ if and only if $A \in (c : brf)_{reg}$ and*

$$\lim_m \sum_{k\in E} |\tilde{a}(m,n,k)| = 0, \text{ uniformly in } n \tag{6.2}$$

for every $E \subset \mathbb{N}$ with natural density zero.

Proof. Let $A \in (st \cap \ell_\infty : brf)_{reg}$. Then the fact that $A \in (c : brf)_{reg}$ immediately follows from the inclusion $c \subset st \cap \ell_\infty$. Now, define a sequence $t = (t_k)$ for $x \in \ell_\infty$ as

$$t_k = \begin{cases} x_k, & k \in E, \\ 0, & k \notin E, \end{cases}$$

where E is any subset of $\mathbb{N}$ with $\delta(E) = 0$. Then $st - \lim t_k = 0$ and $t \in st_0$, so we have that $At \in brf_0$. On the other hand, since $(At)_n = \sum_{k\in E} a_{nk}t_k$, the matrix $B = (b_{nk})$ defined by

$$b_{nk} = \begin{cases} a_{nk}, & k \in E, \\ 0, & k \notin E \end{cases}$$

for all n, must belong to the class (ℓ_∞, brf_0). Hence, the necessity of (6.2) follows from Proposition 5.3.

Conversely, suppose that $A \in (c : brf)_{reg}$ and (6.2) holds. Let $x \in st \cap \ell_\infty$ and $st - \lim x = \ell$. Then, for any given $\varepsilon > 0$ the set $E = \{k : |x_k - \ell| \geq \varepsilon\}$ has density zero and $|x_k - \ell| \leq \varepsilon$ if $k \notin E$. Now, we can write

$$\sum_k \tilde{a}(m,n,k)x_k = \sum_k \tilde{a}(m,n,k)(x_k - \ell) + \ell \sum_k \tilde{a}(m,n,k). \tag{6.3}$$

Since

$$\left|\sum_k \tilde{a}(m,n,k)(x_k - \ell)\right| \leq \|x\| \sum_{k\in E} |\tilde{a}(m,n,k)| + \varepsilon\|A\|,$$

letting $m \to \infty$ in (6.3) and using (6.1) with (6.2), we obtain

$$\lim_m \sum_k \tilde{a}(m,n,k)x_k = \ell.$$

This implies that $A \in (st \bigcap \ell_\infty : brf)_{reg}$ and the proof is completed. □

We can also give some inclusion theorems. First, we need the next lemma.

Lemma 6.5 ([22], Corollary 12)**.** *Let $\mathcal{A} = (a_{mk}(n))$ be a matrix defined by $a_{mk}(n) = \tilde{a}(m,n,k)$ for all $m,n,k \in \mathbb{N}$, satisfying $\|\mathcal{A}\| = \|a_{mk}(n)\| < \infty$ and $\limsup_m \sup_n |a_{mk}(n)| = 0$. Then there exists $y \in \ell_\infty$ with $\|y\| \leq 1$ such that*

$$\limsup_m \sup_n \sum_k \tilde{a}(m,n,k)y_k = \limsup_m \sup_n \sum_k |\tilde{a}(m,n,k)|.$$

Theorem 6.6. *$B_{R\mathscr{B}} - core(Ax) \subseteq \mathcal{K} - core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (c : brf)_{reg}$ and*

$$\limsup_{m} \sup_{n} \sum_{k} |\tilde{a}(m,n,k)| = 1. \tag{6.4}$$

Proof. Let the $B_{R\mathscr{B}} - core(Ax) \subseteq \mathcal{K} - core(x)$ and take $x \in c$ with $\lim x = \ell$. Then, since $\mathcal{K} - core(x) \subseteq \{\ell\}$, $B_{R\mathscr{B}} - core(Ax) \subseteq \{\ell\}$. From here, $brf - \lim Ax = \ell$ which means that $A \in (c : brf)_{reg}$.

Since $A \in (c : brf)_{reg}$, the matrix $\mathcal{A} = \tilde{a}(m,n,k)$ satisfies the conditions of Lemma 6.5. Thus, there exists $y \in \ell_\infty$ with $\|y\| \leq 1$ such that

$$\left\{ w \in \mathbb{C} : |w| \leq \limsup_{m} \sup_{n} \sum_{k} \tilde{a}(m,n,k) y_k \right\} = \left\{ w \in \mathbb{C} : |w| \leq \limsup_{m} \sup_{n} \sum_{k} |\tilde{a}(m,n,k)| \right\}.$$

On the other hand, since $\mathcal{K} - core(y) \subseteq A_1^*(0)$,

$$\left\{ w \in \mathbb{C} : |w| \leq \limsup_{m} \sup_{n} \sum_{k} |\tilde{a}(m,n,k)| \right\} \subseteq A_1^*(0) = \{ w \in \mathbb{C} : |w| \leq 1 \},$$

which implies (6.4).

Conversely, let $w \in B_{R\mathscr{B}} - core(Ax)$. Then, for any given $z \in \mathbb{C}$, we can write

$$\begin{aligned}
|w - z| &\leq \limsup_{m} \sup_{n} |t_{mn}(Ax) - z| \\
&= \limsup_{m} \sup_{n} \left| z - \sum_{k} \tilde{a}(m,n,k) x_k \right| \\
&\leq \limsup_{m} \sup_{n} \left| \sum_{k} \tilde{a}(m,n,k)(z - x_k) \right| + \limsup_{m} \sup_{n} |z| \left| 1 - \sum_{k} \tilde{a}(m,n,k) \right| \\
&= \limsup_{m} \sup_{n} \left| \sum_{k} \tilde{a}(m,n,k)(z - x_k) \right|.
\end{aligned}$$

Now, let $L(x) = \limsup_{k} |x_k - z|$. Then, for any $\varepsilon > 0$, $|x_k - z| \leq L(x) + \varepsilon$ whenever $k \geq k_0$. Hence, we obtain

$$\begin{aligned}
\left| \sum_{k} \tilde{a}(m,n,k)(z - x_k) \right| &= \left| \sum_{k<k_0} \tilde{a}(m,n,k)(z - x_k) + \sum_{k \geq k_0} \tilde{a}(m,n,k)(z - x_k) \right| \\
&\leq \sup_{k} |z - x_k| \sum_{k<k_0} |\tilde{a}(m,n,k)| + [L(x) + \varepsilon] \sum_{k \geq k_0} |\tilde{a}(m,n,k)| \\
&\leq \sup_{k} |z - x_k| \sum_{k<k_0} |\tilde{a}(m,n,k)| + [L(x) + \varepsilon] \sum_{k \geq k_0} |\tilde{a}(m,n,k)|.
\end{aligned}$$

Therefore, applying $\limsup_m \sup_n$ and taking into account the hypothesis, we have

$$|w - z| \leq \limsup_{m} \sup_{n} \left| \sum_{k} \tilde{a}(m,n,k)(z - x_k) \right| \leq L(x) + \varepsilon,$$

which means that $w \in \mathcal{K} - core(x)$. □

The proofs of the following two theorems are entirely analogous to that of Theorem 6.6, so we omit the details.

Theorem 6.7. *$\mathcal{K}-core(Ax) \subseteq B_{R\mathscr{B}}-core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (brf : c)_{reg}$ and* (6.4) *holds.*

Theorem 6.8. *$B_{R\mathscr{B}}-core(Ax) \subseteq B_{R\mathscr{B}}-core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (brf : brf)_{reg}$ and* (6.4) *holds.*

Theorem 6.9. *$B_{R\mathscr{B}}-core(Ax) \subseteq st-core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (st \bigcap \ell_\infty : brf)_{reg}$ and* (6.4) *holds.*

Proof. First of all, we assume that $B_{R\mathscr{B}}-core(Ax) \subseteq st-core(x)$ for all $x \in \ell_\infty$. By taking $x \in st \bigcap \ell_\infty$, one can see that $A \in (st \bigcap \ell_\infty : brf)_{reg}$. Also, since $st-core(x) \subseteq \mathcal{K}-core(x)$ for any x ([20]), the necessity of the condition (6.4) follows from Theorem 6.6.

Conversely, suppose that $A \in (st \bigcap \ell_\infty : brf)_{reg}$ and (6.4) holds, and take $w \in B_{R\mathscr{B}}-core(Ax)$. Now, let $\beta = st-\limsup |z-x_k|$. If we write $E = \{k : |x_k - z| \geq \beta + \varepsilon\}$, then $\delta(E) = 0$ and $|z - x_k| \leq \beta + \varepsilon$ whenever $k \notin E$. Hence we have

$$\begin{aligned}\left|\sum_k \tilde{a}(m,n,k)(z-x_k)\right| &= \left|\sum_{k\in E} \tilde{a}(m,n,k)(z-x_k) + \sum_{k\notin E} \tilde{a}(m,n,k)(z-x_k)\right| \\ &\leq |z-x_k| \sum_{k\in E} |\tilde{a}(m,n,k)| + \sum_{k\notin E} |\tilde{a}(m,n,k)||z-x_k| \\ &\leq |z-x_k| \sum_{k\in E} |\tilde{a}(m,n,k)| + [\beta+\varepsilon] \sum_{k\notin E} |\tilde{a}(m,n,k)|.\end{aligned}$$

By applying the operator $\limsup_m \sup_n$ and using the hypothesis together with (6.2) and (6.4), we obtain that

$$\limsup_m \sup_n \left|\sum_k \tilde{a}(m,n,k)(z-x_k)\right| \leq \beta + \varepsilon. \tag{6.5}$$

Thus, (6.5) implies that $|w - z| \leq \beta + \varepsilon$. Since ε is arbitrary, this means that $w \in st-core(x)$, which completes the proof. □

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