Dual Split Quaternions and Chasles' Theorem in 3-Dimensional Minkowski Space $${\mathbb{E}^{3}_{1}}$$ E 1 3

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Dual Split Quaternions and Chasles' Theorem in 3-Dimensional Minkowski Space $\mathbb{E}_1^3$

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Abstract. In 1831, Michel Chasles proved the existence of a fixed line under a general displacement in $\mathbb{R}^3$. The fixed line called the screw axis of displacement was obtained by McCharthy in [10]. The purpose of this paper is to develop the method which is given for the pure rotation in [14], and thus to obtain the screw axis of spatial displacement in 3-dimensional Minkowski space. Firstly, we give a relation between dual vectors and lines in $\mathbb{E}_1^3$, characterize the screw axis. Also, we discuss the dual split quaternion representation of a spatial displacement.

Keywords. Chasles' theorem, Cayley's formula, screw motion, dual split quaternion, dual semi orthogonal matrix.

1. Introduction

A general body motion is composed of a rotation and translation in three dimensional spaces, briefly is called spatial displacement. In special case, the displacement consists of a pure rotation. In 1775, existence of a fixed line under the rotation is proved by L. Euler in 3-dimensional Euclidean space [4]. The fixed line called *the axis of rotation* passes through the origin and its direction vector is the eigen vector corresponding to the eigen value $\lambda = 1$. After the proof, how to obtain the rotation axis is the significant problem in the world of mathematics. Until today, to solve the problem some methods are given in Euclidean space. Most of the methods are based on Cayley's formula and Rodrigues' rotation formula.

In general case, there is no fixed point under the spatial displacement $X = Ax + d$ with the rotation matrix A, $\det A = 1$ and translation vector $d \in \mathbb{R}^3$, since no inverse of the matrix $A - I$. In 1831, Chasles' theorem shows that a unique line having the same direction vector with rotation axis

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keeps its location before and after the displacement [3]. Each displacement has the fixed line called *the screw axis* in 3-dimensional Euclidean space. Any point on the screw axis moves along the axis and the screw axis need not to pass through the origin. The way for obtaining the axis is discussed by authors Bottema and Roth [2] and McCharthy [10]. Moreover, Chasles' theorem means that a spatial displacement is defined by a rotation about a line (screw axis) and a translation along the same line, called *screw motion.*

In kinematics, space motions with their invariants and characterizations by quaternions, dual quaternions, matrices etc. have large application and research areas. One of them, screw theory, which is the result of Chasles' theorem is investigated largely and a good application of dual constructions [5, 6, 7, 10, 15, 16]. Screw motion is represented by 3×3 dual matrices by McCharthy [10] and Yang [16]. Also, dual forms of angle and screw axis are obtained from the 3×3 dual matrix representation of screw motion by Fischer [5] in 3-dimensional Euclidean space. In recent years, screw motion is investigated in 3-dimensional Minkowski space by authors Özkaldı and Gündoğan [13] and Aydoğmuş, Kula and Yaylı [1, 9]. In these studies, screw motion is represented by dual split quaternions and screw axis is obtained from Cayley's formula.

In our previous paper [14], we showed a new method to obtain the axis of rotation with the help of its semi skew-symmetric part in 3-dimensional Minkowski space. The purpose of this study is to develop the method for the general displacement and obtain the fixed line, screw axis. For implementation of the method, firstly the spatial displacement is represented by a 3×3 dual matrix, then the dual vector representation of screw axis is obtained according to the type of rotation axis, spacelike or timelike vector. Afterwards, depending on the type of rotation axis, the point on the axis is found and the screw axis is located. Later, with the help of the dual form of screw axis, we give dual split quaternionic representation of general displacement. Finally, we add the conclusion to mention about advantages of the given method for the screw axis.

2. Preliminaries

The Minkowski space is the metric space $\mathbb{E}_1^3 = (\mathbb{R}^3, \langle , \rangle)$, where the metric $\langle , \rangle$ is given by

$$\langle u, v\rangle = -u_1v_1 + u_2v_2 + u_3v_3 \tag{2.1}$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. And its matrix form is

$$\langle u, v\rangle = u^t \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} v := u^t \delta v \tag{2.2}$$

where the matrix δ is the sign matrix of Minkowski space. The non-degenerate metric is $\langle , \rangle$ called as the Minkowski metric. The norm of the vector u is

$\sqrt{|\langle u, u\rangle|}$, denoted by $\|u\|$. The Minkowski cross product of the vectors u and v is

$$u \times v = \begin{vmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1). \quad (2.3)$$

Definition 1. A vector $v \in \mathbb{E}_1^3$ is called

1. spacelike if $\langle v, v\rangle > 0$ or $v = 0$,
2. timelike if $\langle v, v\rangle < 0$,
3. lighlike if $\langle v, v\rangle = 0$ and $v \neq 0$.

Definition 2. A 3×3 matrix S is called

1. semi symmetric if $S^T = \delta S\delta$ or $S = \delta S^T\delta$,
2. semi skew-symmetric if $S^T = -\delta S\delta$ or $S = -\delta S^T\delta$,
3. semi orthogonal $S^T = \delta S^{-1}\delta$ or $S^{-1} = \delta S^T\delta$

where δ is the sign matrix of Minkowski space.

(See [11].)

In 3-dimensional Minkowski space, the set of all 3×3 semi orthogonal matrices is a lie group, denoted by $O_1(3,\mathbb{R})$. Also the special subset of $O_1(3,\mathbb{R})$ whose determinant $+1$ determines spatial rotations. On the other hand, the lie algebra of $O_1(3,\mathbb{R})$ is the set of all 3×3 semi skew-symmetric matrices, denoted by $\lozenge_1(3)$. Moreover, every 3×3 semi skew-symmetric matrix determines a vector under the isomorphism Φ that is

$$\Phi : \lozenge_1(3) \to \mathbb{E}_1^3$$
$$S = \begin{pmatrix} 0 & v_3 & -v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \to \Phi(S) = S^v = (v_1, v_2, v_3) \quad (2.4)$$

where for any $w \in \mathbb{E}_1^3$, $S(w) = S^v \times w$ is satisfied.

Definition 3. Let A be a semi orthogonal matrix with determinant $+1$ and $d = (d_1, d_2, d_3)$ is a vector in $\mathbb{E}_1^3$. The transformation $X = Ax + d$ is called a *displacement* in Minkowski space, where $x \in \mathbb{E}_1^3$.

Let show characteristics of the displacement according to value of d in two cases.

Case 1. *If the vector d is zero, the displacement is composed of a pure rotation and it fixed the line called rotation axis of A. Image of any point on the axis is itself. The easy way to find the axis is given in* [14]. *The axis of rotation A is image of its semi skew-symmetric part under the isomorphism* Φ, $\Phi\left(\frac{\hat{A}-\hat{A}^{-1}}{2}\right)$. *If* $A = A^{-1}$, *then the axis of rotation is a non-zero column of the matrix* $A + I_3$.

Case 2. *If the vector d is different from zero, under the displacement* $Ax + d$ *there is no fixed point because* $\lambda = 1$ *is eigen value of the matrix A and the inverse matrix of* $A - I$ *does not exist. Although there is no fixed points,* *there is a fixed line called the screw axis, that has the same position in space before and after the displacement. Any point on this line moves along on it. Direction of the screw axis is the same as the axis of rotation A. The screw axis does not pass over origin, unlike the rotation axis. If the point c is on the screw axis, then the parametric equation of screw axis is*

$$L = c + \lambda b \tag{2.5}$$

where b is the axis of rotation A.

With the screw axis L, the displacement of $x \in \mathbb{E}_1^3$ is explained from a different perspective, called screw motion. The screw motion of x is composed of a rotation of cx about the screw axis L and a translation $d - d^*$ along L, where d^* is the orthogonal part of d to b, see Fig.1.

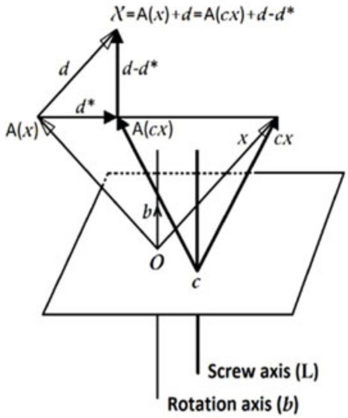

FIGURE 1. Screw motion of the vector x

In this study, we give the way to find the screw axis of displacements with using dual representations of displacements and lines. Now, let us introduce dual numbers and the algebra of dual vectors in 3-dimensional Minkowski space.

Dual numbers have the form $x + \varepsilon y$, where $x, y \in \mathbb{R}$ are called the real and dual part of the dual number, respectively. ε is called the dual unit which satisfies $\varepsilon^2 = 0$. The set of dual numbers has the ring form, denoted by $\mathbb{D}$. If $\alpha = x_1 + \varepsilon y_1$, $\beta = x_2 + \varepsilon y_2 \in \mathbb{D}$, then the operations of addition and multiplication are defined

$$\begin{aligned} \alpha + \beta &= x_1 + x_2 + \varepsilon(y_1 + y_2) \\ \alpha \cdot \beta &= x_1 x_2 + \varepsilon(x_1 y_2 + x_2 y_1) \end{aligned}$$

Any dual matrix $\hat{A} = [\hat{a}_{ij}]$, $\hat{a}_{ij} = a_{ij} + \varepsilon a_{ij}^* \in \mathbb{D}$ is written as $\hat{A} = A + \varepsilon A^*$, where $A = [a_{ij}]$ and $A^* = [a_{ij}^*]$ are real matrices. If $\hat{A}$ satisfies

$\hat{A}^T = \delta\hat{A}^{-1}\delta$, then it is called semi dual orthogonal matrix, where δ is the sign matrix of 3-dimensional Minkowski space. And the set of all semi dual orthogonal matrix is denoted by $O_1(3,\mathbb{D})$. Moreover, if $\hat{A} = A + \varepsilon A^*$ is a semi dual orthogonal matrix where $A \in O_1(3,\mathbb{R})$, with determinant $+1$, then $\hat{A}$ represents a displacement.

The set of dual vectors is formed all pairs of vectors in $\mathbb{E}_1^3$, denoted by $\mathbb{D}_1^3$.

$$\mathbb{D}_1^3 = \left\{ x + \varepsilon x^* |\ x,\ x^* \in \mathbb{E}_1^3,\ \varepsilon^2 = 0 \right\}$$

where ε is the dual unit . If x is spacelike,timelike or lightlike, then the dual vector is called spacelike,timelike or lightlike, respectively. If $\alpha = x + \varepsilon x^*$, $\beta = y + \varepsilon y^* \in \mathbb{D}_1^3$, the inner product of dual vectors is

$$\langle \alpha, \beta \rangle_{\mathcal{D}} = \langle x, y \rangle + \varepsilon(\langle x, y^* \rangle + \langle x^*, y \rangle)$$

where $x, x^*, y, y^* \in \mathbb{E}_1^3$ and $\langle , \rangle$ is the Minkowski inner product. Norm of the dual vector α is $\|\alpha\|_{\mathcal{D}} = \sqrt{\langle \alpha, \alpha \rangle_{\mathcal{D}}}$. If $\|\alpha\|_{\mathcal{D}} = 1$, $\alpha = x + \varepsilon x^*$ is called unit dual vector and satisfies $\| x \| = 1$ and $\langle x, x^* \rangle = 0$.

Theorem 1. [9] *One-to-one correspondence exist between directed timelike or spacelike lines of* $\mathbb{E}_1^3$ *and a unit dual vector* $w + \varepsilon w^* \in \mathbb{D}_1^3$.

i. *If the direction vector is timelike, then the parametric equation of line*

$$w \times w^* + \lambda w \tag{2.6}$$

ii. *If the direction vector is spacelike, then the parametric equation of line*

$$(-w \times w^*) + \lambda w \tag{2.7}$$

where w *and* w^* *are the direction vector and momentum of line, respectively.*

In 3-dimensional Minkowski space, the general displacement $X = Ax + d$ is represented by dual split quaternions in [9]. We give a different characterization for spatial displacements by dual split quaternions in the last part of the article. Now, let us introduce dual split quaternions in $\mathbb{E}_2^4$ whose sign $(-,-,+,+)$.

A dual split quaternion Q is expressed of the form

$$Q = A_0 + A_1 i + A_2 j + A_3 k$$

where $A_0, A_1, A_2, A_3 \in \mathbb{D}$ and i, j, k are split quaternionic units which satisfy the equations

$$i^2 = -1,\ j^2 = 1,\ k^2 = 1$$

and

$$ij = -ji = k, jk = -kj = -i, ki = -ik = j$$

The conjugate of Q is $\bar{Q} = A_0 - A_1 i - A_2 j - A_3 k$ and norm of Q is $N_Q = Q\bar{Q} = A_0^2 + A_1^2 - A_2^2 - A_3^2$. If $N_Q = 1$, then the quaternion Q is called unit dual split quaternion.

A dual split quaternion Q has two part S_Q, V_Q called dual part and vector part of the quaternion, respectively.

$$Q = S_Q + V_Q$$

where $S_Q = A_0 \in \mathbb{D}$ and $V_Q = A_1 i + A_2 j + A_3 k \in \mathbb{D}_1^3$. Also, if $A_1 = a_1 + \varepsilon a_1^*, A_2 = a_2 + \varepsilon a_2^*$ and $A_3 = a_3 + \varepsilon a_3^*$, then

$$V_Q = V_q + \varepsilon V_q^* = (a_1, a_2, a_3) + \varepsilon(a_1^*, a_2^*, a_3^*)$$

where $V_q, V_q^* \in \mathbb{E}_1^3$.

Definition 4. [9] Let $Q = S_Q + V_Q$ be a dual split quaternion and $V_Q = V_q + \varepsilon V_q^*$ be vector part of Q.

i. If $\langle V_q, V_q \rangle > 0$ or $V_q = 0$, V_Q is called spacelike dual split vector.
ii. If $\langle V_q, V_q \rangle < 0$, V_Q is called timelike dual split vector.
iii. If $\langle V_q, V_q \rangle = 0$ and $V_q \neq 0$, V_Q is called lightlike dual split vector where $V_q, V_q^* \in \mathbb{E}_1^3$.

3. On the Screw Axis of Displacement

In this section, we give a useful method to obtain the screw axis in 3-dimensional Minkowski space. Also, to determine the position of screw axis, a specific point on the screw axis is obtained in two ways and some important conclusions are given about that in remark.

To determine the screw axis' position in spatial displacement which consist of rotation A and translation d, needed to find its direction vector and a point on it. The direction of screw axis is the same as the axis of rotation A. While finding the point, the equation

$$Ac + d^* = c \tag{3.1}$$

is used, where d^* is projection of the translation vector d onto a plane perpendicular to the axis of rotation [13].

Before obtaining of the point c by Cayley formula of A, we give an important theorem about Cayley's formula in 3-dimensional Minkowski space. In [12], it is investigated that every semi skew-symmetric matrix defines a semi orthogonal matrix. To verify this idea we give the following theorem.

Theorem 2. *Every B semi skew-symmetric matrix with $\langle \Phi(B), \Phi(B) \rangle \neq 1$ defines a semi orthogonal matrix A as*

$$A = (I - B)^{-1}(I + B) \tag{3.2}$$

where Φ is the isomorphism in (2.4). *This is called the Cayley formula of semi orthogonal matrix A.*

Proof. Let A be a 3×3 semi orthogonal matrix, $x \in \mathbb{E}_1^3$ and $Ax = X$. It is known that the distance between two points is kept under the rotations.

$$\langle x, x \rangle = \langle X, X \rangle$$

And it can be written in the form

$$\langle X - x, X + x \rangle = 0 \tag{3.3}$$

On the other hand,

$$X + x = [A + I]x \text{ and } X - x = [A - I]x \tag{3.4}$$

from the equations (3.4)

$$X - x = [A - I][A + I]^{-1}(X + x) \tag{3.5}$$

is obtained, where $A + I$ is regular except for the case $Ax = -x$.

The matrix $[A - I][A + I]^{-1} = B$ satisfies the property for a general vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $B\alpha$ is orthogonal to α, as specified by (3.3).

$$\begin{aligned} \langle B\alpha, \alpha\rangle &= 0 \\ -\alpha_1 \sum b_{1j}\alpha_j + \sum \alpha_{i+1}\left(\sum b_{(i+1)j}\alpha_j\right) &= 0 \end{aligned} \tag{3.6}$$

where the matrix $B = [b_{ij}]$, $1 \leq i, j \leq 3$.

To satisfy the equation (3.6) elements of B have to be

$$b_{ii} = 0,\ b_{1j} = b_{j1},\ b_{(i+1)j} = -b_{j(i+1)}$$

This means that the matrix B is a semi skew-symmetric matrix. From the equation $B = [A - I][A + I]^{-1}$, Cayley's formula for the semi orthogonal matrix A is obtained

$$A = (I - B)^{-1}(I + B) \text{ or } A = (I + B)(I - B)^{-1} \tag{3.7}$$

where the inverse of $I - B$ exist except for $\langle\Phi(B), \Phi(B)\rangle \neq 1$, because of $\det(I - B) = 1 - (-b_{23}^2 + b_{12}^2 + b_{13}^2) = 1 - \langle\Phi(B), \Phi(B)\rangle$. $\square$

In fact, in the equation (3.2), the matrix B is the semi skew-symmetric matrix form of rotation axis of A. Now, let show that how can be obtained the point c according to the axis of rotation A with its Cayley formula.

Theorem 3. *Let $X = Ax + d$ be a displacement in 3-dimensional Minkowski space, where A is rotation with the timelike axis b and d is translation vector. The point on the screw axis is*

$$c = \frac{1}{2}\left(d^* - \frac{b \times d^*}{\langle b, b\rangle}\right) \tag{3.8}$$

where the vector d^* is projection of the translation vector d onto a plane perpendicular to the rotation axis b.

Proof. To find the point c, the equation has to be solved in (3.1). The equation is

$$(I - A)c = d^*$$

where $I - A = -2(I - B)^{-1}B$ is written from Cayley formula of the semi orthogonal matrix A and B is the semi skew-symmetric matrix form of the rotation axis b.

$$\begin{aligned} Bc &= \frac{-1}{2}(I-B)d^* \\ b\times c &= \frac{-1}{2}(d^*-b\times d^*) \\ c &= \frac{1}{2}\left(d^*-\frac{b\times d^*}{\langle b,b\rangle}\right) \end{aligned}$$

where $\langle b,d^*\rangle=\langle b,c\rangle=0$. $\square$

Remark 1. If the rotation axis b is spacelike, there are some problems so we can't find formula for the point c by this way. Firstly, the formula is not obtained for $\| b \|=1$, because of $\det(I-B)=1-\| b \|^2$ in the Cayley formula of A. And the projection vector d^* could be null when the axis b is spacelike, then it is not certain that the point c is true under the formula (3.8) (see Example 1).

To solve these problems and give general way to find the screw axis of displacement, firstly we obtain dual matrix form of any displacement.

Theorem 4. *Any dual matrix $\hat{A}=A+\varepsilon A^*$ belongs to $O_1(3,\mathbb{D})$ if and only if $A\in O_1(3,\mathbb{R})$ and $A^*=DA$, where $D\in\lozenge_1(3)$.*

Proof. We show that if $\tilde{A}=A+\varepsilon DA\in O_1(3,\mathbb{D})$, then $A\in O_1(3,\mathbb{R})$ and $D\in\lozenge_1(3)$.

$$\tilde{A}(\delta\tilde{A}^T\delta)=I \tag{3.9}$$

$$\tilde{A}(\delta\tilde{A}^T\delta)=A(\delta A^T\delta)+\varepsilon[A(\delta A^T\delta)(\delta D^T\delta)+DA(\delta A^T\delta)] \tag{3.10}$$

from the equality of two dual matrices in (3.9) and (3.10)

$$\delta A^T\delta=A^{-1} \text{ and } \delta D^T\delta=-D$$

So, $A\in O_1(3,\mathbb{R})$ and $D\in\lozenge_1(3)$.

On the other hand, if $\hat{A}=A+\varepsilon A^*\in O_1(3,\mathbb{D})$, it is sufficient to put $D=A^*A^{-1}$. Then, the expression of the displacement turns $\hat{A}=A+\varepsilon DA$. $\square$

Now, we give following theorems to get the screw axis by using dual matrix representation of the displacement. Thus, we obtained dual representation of the screw axis.

Theorem 5. *Let $X=Ax+d$ be a displacement in 3-dimensional Minkowski space and its dual matrix representation be $\hat{A}=A+\varepsilon DA$, the dual vector $\Phi\left(\frac{\hat{A}-\hat{A}^{-1}}{2}\right)$ is invariant under the displacement $\hat{A}$. So the screw axis of $\hat{A}$ is*

$$w+\varepsilon w^*=\frac{\Phi\left(\frac{\hat{A}-\hat{A}^{-1}}{2}\right)}{\|\Phi\left(\frac{\hat{A}-\hat{A}^{-1}}{2}\right)\|_{\mathcal{D}}} \tag{3.11}$$

where the transformation Φ is the isomorphism in (2.4).

Proof. Let the matrix $\hat{S}$ be the skew-symmetric part of $\hat{A}$.

$$\begin{aligned} \hat{A}\hat{S}\hat{A}^{-1} &= \hat{A}\left(\tfrac{\hat{A}-\hat{A}^{-1}}{2}\right)\hat{A}^{-1} \\ &= \tfrac{\hat{A}-\hat{A}^{-1}}{2} \\ &= \hat{S} \end{aligned} \tag{3.12}$$

so the matrix $\hat{S}$ is invariant under the product $\hat{A}\hat{S}\hat{A}^{-1}$. And under the isomorphism Φ, $(\hat{A}\hat{S}\hat{A}^{-1})^v = \hat{A}\hat{S}^v$, also from the equation (3.12), $\Phi(\hat{A}\hat{S}\hat{A}^{-1}) = \Phi(\hat{S}) = \hat{S}^v$. This shows that $\hat{S}^v = \Phi\left(\frac{\hat{A}-\hat{A}^{-1}}{2}\right)$ is invariant under the displacement $\hat{A}$. On the other hand, the matrix $\hat{S}$ is a dual semi skew-symmetric matrix, so under the isomorphism Φ its image is a dual vector

$$\Phi(\hat{S}) = \Phi\left(\frac{\hat{A}-\hat{A}^{-1}}{2}\right) = \Phi\left(\frac{A-A^{-1}}{2}\right) + \varepsilon\Phi\left(\frac{DA-(DA)^{-1}}{2}\right) = b + \varepsilon b^* \tag{3.13}$$

where $b, b^* \in \mathbb{E}_1^3$. The unit dual vector

$$w + \varepsilon w^* = \frac{b+\varepsilon b^*}{\| b+\varepsilon b^* \|_{\mathcal{D}}} \tag{3.14}$$

represents the screw axis of displacement $\hat{A}$. Where the unit vector w is the axis of rotation A. If the axis w is timelike or spacelike vector, then the point c on the screw axis is

$$w \times w^* \text{ or } -w \times w^* \tag{3.15}$$

respectively. □

Let's see that when the rotation axis is timelike, the point in (3.15) satisfies the same equality with the point in (3.8) which is obtained from the formula.

If a unit timelike dual vector $w+\varepsilon w^*$ is invariant under the displacement $\hat{A} = A + \varepsilon DA$, then

$$\begin{aligned} (A+\varepsilon DA)(w+\varepsilon w^*) &= w + \varepsilon w^* \\ Aw + \varepsilon(Aw^* + DAw) &= w + \varepsilon w^* \end{aligned} \tag{3.16}$$

from equality of two dual vectors in (3.16) $Aw = w$, so it is easily seen that w is the axis of rotation A. Also,

$$\begin{aligned} Aw^* + DAw &= w^* \\ (I-A)w^* &= DAw = Dw \end{aligned} \tag{3.17}$$

with the help of the Cayley formula of the orthogonal matrix A, $I - A = -2(I-W)^{-1}W$, where the semi skew-symmetric matrix W is the image of unit timelike vector w under the isomorphism Φ.

$$\begin{aligned} W(w^*) &= -\frac{1}{2}\left[(I-W)Dw\right] \\ w \times w^* &= -\frac{1}{2}\left[d \times w + \langle w, w\rangle d - \langle w, d\rangle w\right] \end{aligned}$$

$$w^* = \frac{1}{2}\left[w \times (d \times w) + d \times w\right] \tag{3.18}$$

On the other hand, the projection of d onto the plane which is perpendicular to the rotation axis w is the vector d^*,then the equation of d^* is

$$d^* = d + \langle d, w\rangle w$$

so, $d^* \times w = d \times w$ and the equation (3.18) turns

$$w^* = \frac{1}{2}(d^* + d^* \times w)$$

The point c is

$$\begin{aligned} c &= w \times w^* \\ &= \tfrac{1}{2}(d^* + w \times d^*) \end{aligned} \tag{3.19}$$

so, two expressions of the point c, (3.8) and (3.19) are equal.

If the dual matrix form of displacement $\hat{A}$ is equal its inverse matrix, then the theorem 5 does not work for finding the screw axis of displacement. So, we give the following theorem to get the axis for this condition.

Theorem 6. *Let $\hat{A} = A + \varepsilon DA$ be a displacement in 3-dimensional Minkowski space. If $\hat{A} = \hat{A}^{-1}$, then the non-zero column of the matrix $\hat{A} + I$ is the screw axis of the displacement.*

Proof. If $\hat{A}$ is a dual semi orthogonal matrix which satisfies $\hat{A} = \hat{A}^{-1}$, then $\hat{A}^2 = I$. $\hat{A}(\hat{A} + I) = \hat{A} + I$, so $\hat{A} + I$ is invariant under the displacement $\hat{A}$. So, columns of $\hat{A} + I$ are fixed under the displacement $\hat{A}$. On the other hand, $rank(\hat{A} - I) + rank(\hat{A} + I) = 3$ and $rank(\hat{A} - I) = 2$,so $rank(\hat{A} + I) = 1$ [8]. Thus, there is a non-zero column of $\hat{A}$ and the non-zero column is the screw axis of $\hat{A}$. $\square$

Example 1. If the displacement is $\hat{A} = A + \varepsilon DA = \begin{pmatrix} \frac{5}{4} + \frac{3}{4}\varepsilon & \frac{3}{4} + \frac{5}{4}\varepsilon & -\varepsilon \\ \frac{3}{4} + \frac{5}{4}\varepsilon & \frac{5}{4} + \frac{3}{4}\varepsilon & -\varepsilon \\ \frac{-1}{2}\varepsilon & \frac{1}{2}\varepsilon & 1 \end{pmatrix}$, where its rotation part is $A = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{5}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and matrix form of translation vector $d = (1, 1, 1)$ is $D = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. The axis of rotation A is $b = \Phi\left(\frac{A - A^{-1}}{2}\right) = \left(0, 0, \frac{3}{4}\right)$.

Firstly, we use the formula (3.8) to find the point c

$$c = \frac{1}{2}\left(d^* - \frac{b \times d^*}{\langle b, b\rangle}\right) = (0, 0, 0)$$

where the vector $d^* = (1,1,0)$ is the orthogonal part of d to b. But the point $c = (0,0,0)$ does not satisfy the equation in (3.1). On the other hand, with the help of theorem 5 it can be solved. The screw axis of displacement is

$$w + \varepsilon w^* = \frac{\left(\frac{3}{4}\varepsilon, \frac{3}{4}\varepsilon, \frac{3}{4} + \frac{5}{4}\varepsilon\right)}{\parallel \left(\frac{3}{4}\varepsilon, \frac{3}{4}\varepsilon, \frac{3}{4} + \frac{5}{4}\varepsilon\right) \parallel_{\mathcal{D}}} = (\varepsilon, \varepsilon, 1)$$

$w = (0,0,1)$ is the unit spacelike direction vector of displacement, so the point on the axis $c = -w \times w^* = (-1,-1,0)$ satisfies the equation (3.1).

Example 2. If a displacement $\hat{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} - \frac{\varepsilon}{\sqrt{2}} & \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} & \frac{-1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \end{pmatrix}$, whose rotation part is $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$ and translation vector is $d = (1,0,0)$ satisfies $\hat{A} = \hat{A}^{-1}$, then the screw axis is the non-zero column of the matrix $\hat{A} + I$.

$$\begin{aligned} \hat{A} + I &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 + \frac{1}{\sqrt{2}} - \frac{\varepsilon}{\sqrt{2}} & \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} & 1 - \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & [\sqrt{2} + 1 - (\sqrt{2}+2)\varepsilon]\left(\frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}}\right) & \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \\ 0 & [\sqrt{2} + 1 - (\sqrt{2}+2)\varepsilon]\left(1 - \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}}\right) & 1 - \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

so the screw axis of $\hat{A}$ is unit form of the vector $\left(0, \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}}\right)$

$$w + \varepsilon w^* = \frac{1}{\sqrt{4 - 2\sqrt{2}}}\left(0, 1 + \frac{1 - \sqrt{2}}{2}\varepsilon, \sqrt{2} - 1 + \frac{1}{2}\varepsilon\right)$$

where the direction vector $w = \left(0, \frac{1}{\sqrt{4-2\sqrt{2}}}, \frac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}}\right)$ is spacelike, the point on the screw axis is $\left(\frac{1}{2}, 0, 0\right)$.

4. Dual Split Quaternion Representation

In this section, we show that a spatial displacement with the screw axis L is represented by dual split quaternions. Also the dual matrix form of displacement is characterized by dual and vector parts of the dual split quaternion which represents the displacement.

To show the displacement representation, firstly we define an isomorphism between 3- dimensional dual vector space $\mathbb{D}_1^3$ and $\lozenge_1(3)$ which is the vector space of 3×3 semi skew-symmetric matrices .

$$\begin{gathered}\Psi:\mathbb{D}_1^3\to\lozenge_1(3)\\ L=(L_1,L_2,L_3)\to\Psi(L)=\begin{pmatrix}0 & L_3 & -L_2\\ L_3 & 0 & -L_1\\ -L_2 & L_1 & 0\end{pmatrix}.\end{gathered}\tag{4.1}$$

Let $Q=S_Q+V_Q$ be a dual split quaternion with the dual part $S_Q=A_0\in\mathbb{D}$ and the vector part $V_Q=A_1i+A_2j+A_3k\in\mathbb{D}_1^3$ where $A_0=a_0+\varepsilon a_0^*$, $A_1=a_1+\varepsilon a_1^*$, $A_2=a_2+\varepsilon a_2^*$, and $A_3=a_3+\varepsilon a_3^*$.

i. If $V_Q\neq 0$ is a spacelike dual vector, then the dual split quaternion Q is written as

$$Q=\cosh\frac{\phi}{2}+\sinh\frac{\phi}{2}L \tag{4.2}$$

where $\cosh\frac{\phi}{2}=A_0$, $\sinh\frac{\phi}{2}=\sqrt{-A_1^2+A_2^2+A_3^2}$ and $L=\frac{A_1i+A_2j+A_3k}{\sqrt{-A_1^2+A_2^2+A_3^2}}$, $\langle L,L\rangle_{\mathcal{D}}=1$. The angle $\frac{\phi}{2}=\frac{\varphi}{2}+\varepsilon\frac{\varphi^*}{2}$ is a dual angle where $\frac{\varphi}{2}=\operatorname{arccosh}(a_0)$ and $\frac{\varphi^*}{2}=\frac{a_0^*}{\sqrt{a_0^2-1}}$.

The unit dual split quaternion $Q=\cosh\frac{\phi}{2}+\sinh\frac{\phi}{2}L$ represent a displacement as

$$D=I_3+\sinh\phi[L]+(-1+\cosh\phi)[L]^2 \tag{4.3}$$

where $\phi=\varphi+\varepsilon\varphi^*$ is the dual angle, φ is the angle of displacement's rotation part and $[L]=\Psi(L)$, L is the unit dual vector form of screw axis [9].

On the other hand, the dual and vector parts of Q are $S_Q=\cosh\frac{\phi}{2}$ and $V_Q=\sinh\frac{\phi}{2}L$, respectively. So,

$$\begin{aligned}\sinh\phi[L] &= 2\sinh\frac{\phi}{2}\cosh\frac{\phi}{2}[L]=2S_Q[V_Q]\\ (-1+\cosh\phi)[L]^2 &= 2\sinh^2\frac{\phi}{2}[L]^2=2[V_Q]^2\end{aligned}\tag{4.4}$$

From (4.4), the equation of the displacement is obtained in terms of the dual split quaternion Q, as

$$D=I_3+2S_Q[V_Q]+2[V_Q]^2 \tag{4.5}$$

where $[V_Q]$ is the image of V_Q under the isomorphism Ψ.

ii. If V_Q is a timelike dual vector, then the dual split quaternion Q is written as

$$Q=\cos\frac{\phi}{2}+\sin\frac{\phi}{2}L \tag{4.6}$$

where $\cos\frac{\phi}{2}=A_0$, $\sin\frac{\phi}{2}=\sqrt{A_1^2-A_2^2-A_3^2}$ and $L=\frac{A_1i+A_2j+A_3k}{\sqrt{A_1^2-A_2^2-A_3^2}}$, $\langle L,L\rangle_{\mathcal{D}}=-1$. The angle $\frac{\phi}{2}=\frac{\varphi}{2}+\varepsilon\frac{\varphi^*}{2}$ is a dual angle where $\frac{\varphi}{2}=\arccos(a_0)$ and $\frac{\varphi^*}{2}=\frac{-a_0^*}{\sqrt{1-a_0^2}}$.

The unit dual split quaternion $Q = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}L$ represent a displacement as

$$D = I_3 + \sin\phi[L] + (1-\cos\phi)[L]^2 \tag{4.7}$$

where $\phi = \varphi + \varepsilon\varphi^*$ is the dual angle, φ is the angle of displacement's rotation part and $[L] = \Psi(L)$, L is the unit dual vector form of screw axis [9].

From the equations $\sin\phi[L] = 2S_Q[V_Q]$ and $(1-\cos\phi)[L]^2 = 2[V_Q]^2$, the equation of the displacement is obtained in terms of the dual split quaternion Q as

$$D = I_3 + 2S_Q[V_Q] + 2[V_Q]^2 \tag{4.8}$$

where $[V_Q]$ is the image of V_Q under the isomorphism Ψ.

5. Conclusion

Searching of the properties which are invariant under displacement is a significant and efficient area in kinematics, especially the screw axis which is fixed line under a displacement. Existence of the screw axis was proved by M. Chasles in 1831 and today, many articles are created to find it. Most of them are based on Cayley's rotation formula. But, we show that the formula is not enough to obtain the screw axis in 3-dimensional Minkowski space. On the other hand, by the method which is given in this study the screw axis is obtained in all cases. Moreover, the method gives directly the equation of screw axis with its dual representation. With the help of the dual representation of screw axis, its direction vector and position point are found directly, too.

Chasles' theorem says that a displacement can be described as a pure rotation around the screw axis plus a translation along the same axis. In this regard, displacements are similar to dual split quaternions in 3-dimensional Minkowski space. So, an important advantage of the dual characterization of screw axis is seen when giving dual split quaternionic representation for any displacement. Thus, we obtain a good relationship between general rigid body motion group and the lie group of dual split quaternions which is a useful material in CAD and animation systems, robotics, navigation, hand-eye calibration, etc.

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References

[1] O. Aydoğmuş. L.Kula, Y. Yaylı, *On point-line displacement in Minkowski 3-space.* Differ. Geom. Dyn. Syst. **10** (2008), 32–43.

[2] O. Bottema, B. Roth, *Theoretical Kinematics.* North-Holland Press, New York, 1979.

[3] M. Chasles, *Note sur les propriétés générales du système de deux corps semblables entre eux, placés d'une manière quelconque dans l'espace; et sur le déplacement fini, ou infiniment petit d'un corps solide libre*. Bulletin des Sciences Mathématiques Férussac **14** (1831), 321–326.

[4] L. Euler, *Formulae generales pro translatione quacunque corporum rigidorum.* Novi Comm. Acad. Sci. Imp. Petrop. **20** (1776), 189-207.

[5] I.S. Fischer, *The dual angle and axis of a screw motion.* Mech. Mach. Theory Vol.33 (1998), 331-340.

[6] D. Hestenes, *New Tools for Computational Geometry and Rejuvenation of Screw Theory.* Geometric Algebra Computing, Springer (2010), 3-33.

[7] A. Karger, J. Novak, *Space Kinematics and Lie Groups.* Gordon and Breach Science Publishers, New York, London, Paris, Montreux, Tokyo 1985.

[8] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry Vol 1.*. Interscience, 1963.

[9] L. Kula, Y. Yaylı, *Dual split quaternions and screw motion in Minkowski 3-space.* Iran. J. Sci. Technol. Trans. A Sci. **30** (2006), 245–258.

[10] J.M. McCarthy, *An Introduction to Theoretical Kinematics.* MIT Press, Cambridge, 1990.

[11] B. O'Neill, *Semi-Riemannian Geometry.* Pure and Applied Mathematics, 103, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.

[12] S. Özkaldı, H. Gündoğan, *Cayley formula, Euler parameters and rotations in 3-dimensional Lorentzian space.* Adv. Appl. Clifford Algebras **20** (2010), 367-377.

[13] S. Özkaldı, H. Gündoğan, *Dual split quaternions and screw motion in 3-dimensional Lorentzian space.* Adv. Appl. Clifford Algebras **21** (2011), 193-202.

[14] C. Ramis, Y. Yaylı, On the axis of rotation in 3-dimensional Minkowski space, Subscript.

[15] X.Wang, D. Han, C. Yu, Z. Zheng, *The geometric structure of unit dual quaternion with application in kinematic control.* J. Math. Anal. Appl **389** (2012), 1352-1364.

[16] A.T Yang, *Displacement analysis of spatial five link mechanisms using* 3×3 *matrices with dual numbers elements.* ASME J. Eng. for Ind. **1**(1969), 152-157.

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