



# Application of Petrov-Galerkin finite element method to shallow water waves model: Modified Korteweg-de Vries equation

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**Abstract.** In this article, modified Korteweg-de Vries (mKdV) equation is solved numerically by using lumped Petrov-Galerkin approach, where weight functions are quadratic and the element shape functions are cubic B-splines. The proposed numerical scheme is tested by applying four test problems including single solitary wave, interaction of two and three solitary waves, and evolution of solitons with the Gaussian initial condition. In order to show the performance of the algorithm, the error norms,  $L_2$ ,  $L_\infty$ , and a couple of conserved quantities are computed. For the linear stability analysis of numerical algorithm, Fourier method is also investigated. As a result, the computed results show that the presented numerical scheme is a successful numerical technique for solving the mKdV equation. Therefore, the presented method is preferable to some recent numerical methods.

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## 1. Introduction

The theory of nonlinear evolution equations (NLEEs) is a very popular and fascinating area of research in the field of applied mathematics and theoretical physics [1-9]. A few of the focused areas of research with NLEEs are fluid dynamics, nonlinear optics, nuclear physics, plasma physics, mathematical biosciences, and several others. This paper will focus on the numerical aspects of the dynamics of shallow water waves along lakes' shores and beaches modeled by a very popular NLEE. This is the modified Korteweg-de Vries (mKdV) equation [10-21].

While several forms of numerical analyses were reported in the past, this paper addresses mKdV

equation with a very popular and effective numerical scheme. This is the cubic B-spline Petrov-Galerkin method. The topics covered in this paper are linear stability analysis, the dynamics of single solitary wave of mKdV equation, interaction of two solitary waves and three solitary waves, and evolution of solitons. Several numerical examples are given to illustrate this powerful algorithm. In conclusion, the error analysis is also investigated in detail.

## 2. Cubic B-spline Petrov-Galerkin method

In this study, we will consider the modified Korteweg-de Vries (mKdV) equation:

$$U_t + \varepsilon U^2 U_x + \mu U_{xxx} = 0, \quad (1)$$

with the boundary conditions  $U \rightarrow 0$  as  $x \rightarrow \pm\infty$ , where  $\varepsilon$  and  $\mu$  are positive parameters, and subscripts

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$x$  and  $t$  denote the differentiation.

Let us consider the solution domain is limited to a finite interval  $a \leq x \leq b$ . The interval  $[a, b]$  is divided into  $N$  equal subinterval by the points  $x_m$ , such that  $a = x_0 < x_1 \cdots < x_N = b$  and length  $h = \frac{b-a}{N} = (x_{m+1} - x_m)$ ,  $m = 1, 2, \dots, N$ . The mKdV equation (Eq. (1)) is considered with respect to the boundary conditions:

$$\begin{aligned} U(a, t) = 0, \quad U(b, t) = 0, \\ U_x(a, t) = 0, \quad U_x(b, t) = 0, \quad t > 0, \end{aligned} \tag{2}$$

and the initial condition:

$$U(x, 0) = f(x), \quad a \leq x \leq b, \tag{3}$$

where  $f(x)$  is a prescribed function. Physical boundary conditions require  $U$  and  $U_x \rightarrow 0$  that  $U \rightarrow 0$  for  $x \rightarrow \pm\infty$ . The cubic B-spline functions  $\phi_m(x)$ ,  $m = -1(1)N + 1$ , are defined at the knots  $x_m$  by the relationships [22]:

$$\phi_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3 & x \in [x_{m-2}, x_{m-1}], \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3 & x \in [x_{m-1}, x_m], \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3 & x \in [x_m, x_{m+1}], \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}], \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The set of functions  $\{\phi_{-1}, \phi_0, \phi_1, \phi_2\}$  forms a basis for functions defined over the interval  $a \leq x \leq b$ . So, the numerical solution  $U_N(x, t)$  to the exact solution  $U_N(x, t)$  is given by:

$$U_N(x, t) = \sum_{j=-1}^{N+1} \delta_j(t) \phi_j(x), \tag{5}$$

where  $\delta_j$  is time-dependent quantities to be determined from the boundary and weighted residual conditions. Each cubic B-spline covers four elements so that each element  $[x_m, x_{m+1}]$  is covered by four splines. Applying a local coordinate system to typical finite element  $[x_m, x_{m+1}]$  is defined by:

$$h\eta = x - x_m, \tag{6}$$

such that  $0 \leq \eta \leq 1$ , the finite interval  $[x_m, x_{m+1}]$  is

converted to a more easily workable interval  $[0, 1]$ . In this case, the cubic B-spline shape functions depending on variable  $\eta$  over the element  $[0, 1]$  can be defined in the following form:

$$\begin{aligned} \phi_{m-1} &= (1 - \eta)^3, \\ \phi_m &= 1 + 3(1 - \eta) + 3(1 - \eta)^2 - 3(1 - \eta)^3, \\ \phi_{m+1} &= 1 + 3\eta + 3\eta^2 - 3\eta^3, \\ \phi_{m+2} &= \eta^3. \end{aligned} \tag{7}$$

All splines, apart from  $\phi_{m-1}(x)$ ,  $\phi_m(x)$ ,  $\phi_{m+1}(x)$ , and  $\phi_{m+2}(x)$ , are null over the element  $[x_m, x_{m+1}]$ . Over the typical element  $[x_m, x_{m+1}]$ , the numerical solution  $U_N(x, t)$  is given by:

$$U_N(x, t) = \sum_{j=m-1}^{m+2} \phi_j(x) \delta_j(t), \tag{8}$$

where  $\delta_{m-1}, \delta_m, \delta_{m+1}$ , and  $\delta_{m+2}$  act as element parameters and B-splines  $\phi_{m-1}, \phi_m, \phi_{m+1}$ , and  $\phi_{m+2}$  as element shape functions. The values of  $\phi_j(x)$  and its derivative may be tabulated as in Table 1.

Using cubic B-splines (Eq. (7)) and trial function (Eq. (8)), the nodal values of  $U, U'$ , and  $U''$  at the knot  $x_m$  are given in terms of the element parameters  $\delta_m$  by:

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-1} + 4\delta_m + \delta_{m+1}, \\ U'_m &= U'(x_m) = \frac{3}{h}(-\delta_{m-1} + \delta_{m+1}), \\ U''_m &= U''(x_m) = \frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}), \end{aligned} \tag{9}$$

where the symbols ' and '' denote the first and second differentiations with respect to  $x$ , respectively. The splines  $\phi_m(x)$  and their two principle derivatives vanish outside the interval  $[x_{m-2}, x_{m+2}]$ . The weight function  $\psi_m$  is taken a quadratic B-spline. Quadratic B-spline  $\psi_m$  at the knots  $x_m$  is defined over the interval  $[a, b]$  by:

**Table 1.** Cubic B-splines and its derivatives at nodes  $x_m$ .

$x$	$x_{m-2}$	$x_{m-1}$	$x_m$	$x_{m+1}$	$x_{m+2}$
$\phi_m(x)$	0	1	4	1	0
$\phi'_m(x)$	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0
$\phi''_m(x)$	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0

$$\psi_m(x) = \frac{1}{h^2} \begin{cases} (x_{m+2} - x)^2 - 3(x_{m+1} - x)^2 + 3(x_m - x)^2 & [x_{m-1}, x_m], \\ (x_{m+2} - x)^2 - 3(x_{m+1} - x)^2 & [x_m, x_{m+1}], \\ (x_{m+2} - x)^2 & [x_{m+1}, x_{m+2}], \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

Using the local coordinate transformation for the finite element  $[x_m, x_{m+1}]$  by  $h\eta = x - x_m$ ,  $0 \leq \eta \leq 1$ , quadratic B-spline  $\psi_m$  can be defined as:

$$\begin{aligned} \psi_{m-1} &= (1 - \eta)^2, \\ \psi_m &= 1 + 2\eta - 2\eta^2, \\ \psi_{m+1} &= \eta^2. \end{aligned} \quad (11)$$

Applying the Petrov-Galerkin approach to Eq. (1), we obtain the weak form of Eq. (1):

$$\int_0^1 \psi(U_t + \varepsilon U^2 U_x + \mu U_{xxx}) dx = 0. \quad (12)$$

For a single element  $[x_m, x_{m+1}]$ , using transformation in Eq. (6) into Eq. (12), we obtain:

$$\int_0^1 \psi \left( U_t + \varepsilon \left( \frac{U^2}{h} \right) U_\eta + \mu \left( \frac{1}{h^3} \right) U_{\eta\eta\eta} \right) d\eta = 0. \quad (13)$$

Integrating Eq. (13) by parts and using Eq. (1) lead to:

$$\int_0^1 [\psi(U_t + \lambda U_\eta) + \xi \Psi_\eta U_{\eta t}] d\eta = \xi \Psi U_{\eta t}|_0^1, \quad (14)$$

where  $\lambda = \frac{U^2}{h}$  and  $\xi = \frac{\mu}{h^3}$ . Taking the weight function,  $\Psi_i$ , with quadratic B-spline shape functions given by Eq. (11) and substituting approximation in Eq. (8) into integral Eq. (14), we obtain the element contributions in the form:

$$\begin{aligned} \sum_{j=m-1}^{m+2} \left[ \int_0^1 \psi_i \phi_j d\eta \right] \delta_j^e + \sum_{j=m-1}^{m+2} \left[ \left( \varepsilon \lambda \int_0^1 \psi_i \phi_j' d\eta \right) - \left( \xi \int_0^1 \psi_i' \phi_j'' d\eta \right) + (\xi \psi_i \phi_j''|_0^1) \right] \delta_j^e = 0, \end{aligned} \quad (15)$$

which can be written in a matrix form as follows:

$$[A^e] \delta^e + [\varepsilon \lambda B^e - \xi(C^e - D^e)] \delta^e = 0, \quad (16)$$

where  $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$  are the element parameters, and the dot denotes differentiation with

respect to  $t$ . The element matrices  $A^e$ ,  $B^e$ ,  $C^e$ , and  $D^e$  are rectangular  $3 \times 4$  given by the following integrals:

$$A_{ij}^e = \int_0^1 \psi_i \phi_j d\eta = \frac{1}{60} \begin{bmatrix} 10 & 71 & 38 & 1 \\ 19 & 221 & 221 & 19 \\ 1 & 38 & 71 & 10 \end{bmatrix},$$

$$B_{ij}^e = \int_0^1 \psi_i \phi_j' d\eta = \frac{1}{10} \begin{bmatrix} -6 & -7 & 12 & 1 \\ -13 & -41 & 41 & 13 \\ -1 & -12 & 7 & 6 \end{bmatrix},$$

$$C_{ij}^e = \int_0^1 \psi_i' \phi_j'' d\eta = \begin{bmatrix} -4 & 6 & 0 & -2 \\ 2 & -6 & 6 & -2 \\ 2 & 0 & -6 & 4 \end{bmatrix},$$

$$D_{ij}^e = \psi_i \phi_j''|_0^1 = \begin{bmatrix} -6 & 12 & -6 & 0 \\ -6 & 18 & -18 & 6 \\ 0 & 6 & -12 & 6 \end{bmatrix},$$

where  $i$  takes only the values 1, 2, and 3;  $j$  takes only the values  $m - 1$ ,  $m$ ,  $m + 1$ , and  $m + 2$  for the typical element  $[x_m, x_{m+1}]$ . A lumped value for  $\lambda$  is found from  $\frac{1}{4}(U_m + U_{m+1})^2$  as:

$$\lambda = \frac{1}{4h} (\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^2. \quad (17)$$

By considering together contributions from all elements, the following matrix equation becomes:

$$[A] \dot{\delta} + [\varepsilon \lambda B - \xi(C - D)] \delta = 0, \quad (18)$$

where  $\delta = (\delta_{-1}, \delta_0, \dots, \delta_N, \delta_{N+1})^T$  is a global element parameter. The matrices  $A$ ,  $B$ , and  $\lambda D$  are rectangular and  $m$  of each row has the following form:

$$A = \frac{1}{60} (1, 57, 302, 302, 57, 1),$$

$$\begin{aligned} \lambda B = \frac{1}{10} &(-\lambda_1, -12\lambda_1 - 13\lambda_2, 7\lambda_1 - 41\lambda_2 \\ &- 6\lambda_3, 6\lambda_1 + 41\lambda_2 - 7\lambda_3, 13\lambda_2 + 12\lambda_3, \lambda_3), \end{aligned}$$

$$C = 2(1, 1, -8, 8, -1, -1),$$

$$D = (0, 0, 0, 0, 0, 0),$$

where:

$$\lambda_1 = \frac{1}{4h} (\delta_{m-2} + 5\delta_{m-1} + 5\delta_m + \delta_{m+1})^2,$$

$$\lambda_2 = \frac{1}{4h} (\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^2,$$

$$\lambda_3 = \frac{1}{4h} (\delta_m + 5\delta_{m+1} + 5\delta_{m+2} + \delta_{m+3})^2.$$

Substituting the Crank-Nicolson approach,  $\delta = \frac{1}{2}(\delta^n + \delta^{n+1})$ , and the forward finite difference,  $\dot{\delta} = \frac{\delta^{n+1} - \delta^n}{\Delta t}$ , in Eq. (18), we obtain the following matrix system:

$$\left\{ A + [\varepsilon\lambda B - \xi(C - D)] \frac{\Delta t}{2} \right\} \delta^{n+1} = \left\{ A - [\varepsilon\lambda B - \xi(C - D)] \frac{\Delta t}{2} \right\} \delta^n, \quad (19)$$

where  $\Delta t$  is the time step. Applying the boundary condition (2) to the system (19), we make the matrix equation square. The resulting system can be efficiently solved by a variant of the Thomas algorithm. Two or three inner iterations are applied to  $\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$  at each time in order to improve the accuracy. A typical member of the matrix system (19) may be written in terms of the nodal parameters  $\delta^n$  and  $\delta^{n+1}$  as:

$$\begin{aligned} \gamma_1 \delta_{m-2}^{n+1} + \gamma_2 \delta_{m-1}^{n+1} + \gamma_3 \delta_m^{n+1} + \gamma_4 \delta_{m+1}^{n+1} + \gamma_5 \delta_{m+2}^{n+1} \\ + \gamma_6 \delta_{m+3}^{n+1} = \gamma_6 \delta_{m-2}^n + \gamma_5 \delta_{m-1}^n + \gamma_4 \delta_m^n \\ + \gamma_3 \delta_{m+1}^n + \gamma_2 \delta_{m+2}^n + \gamma_1 \delta_{m+3}^n, \end{aligned} \quad (20)$$

where:

$$\begin{aligned} \gamma_1 &= \frac{1}{60} - \frac{\varepsilon\lambda\Delta t}{20} - \xi\Delta t, \\ \gamma_2 &= \frac{57}{60} - \frac{25\varepsilon\lambda\Delta t}{20} - \xi\Delta t, \\ \gamma_3 &= \frac{302}{60} - \frac{40\varepsilon\lambda\Delta t}{20} + 8\xi\Delta t, \\ \gamma_4 &= \frac{302}{60} + \frac{40\varepsilon\lambda\Delta t}{20} - 8\xi\Delta t, \\ \gamma_5 &= \frac{57}{60} + \frac{25\varepsilon\lambda\Delta t}{20} + \xi\Delta t, \\ \gamma_6 &= \frac{1}{60} + \frac{\varepsilon\lambda\Delta t}{20} + \xi\Delta t, \end{aligned}$$

which all depend on  $\delta^n$ .

To evaluate the vector parameters,  $\delta^n$ , the initial vector  $\delta^0$  must be determined from the initial and boundary conditions. So, the approximation in Eq. (8) can be rewritten for the initial condition:

$$U_N(x, 0) = \sum_{m=-1}^{N+1} \phi_m(x) \delta_m^0, \quad (21)$$

where the parameters  $\delta_m^0$  will be determined. Using relations at the knots:

$$\begin{aligned} U_N(x_m, 0) &= U(x_m, 0), \\ U'_N(x_0, 0) &= U'(x_N, 0) = 0, \quad m = 0, 1, \dots, N, \end{aligned} \quad (22)$$

together with derivative condition, the initial vector  $\delta^0$  can be determined from the following matrix equation:

$$\begin{bmatrix} -3 & 0 & 3 \\ 1 & 4 & 1 \\ & & \ddots \\ & & & 1 & 4 & 1 \\ & & & -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} \delta_{-1}^0 \\ \delta_0^0 \\ \vdots \\ \delta_N^0 \\ \delta_{N+1}^0 \end{bmatrix} = \begin{bmatrix} U'(x_{0,0}) \\ U(x_{0,0}) \\ \vdots \\ U(x_{N,0}) \\ U'(x_{N,0}) \end{bmatrix},$$

which can be solved using a variant of the Thomas algorithm.

### 3. Stability analysis

For the linear stability analysis of the numerical algorithm, we use the Fourier method and assume that the quantity  $U$  in the non-linear term  $U^2 U_x$  is locally constant. In this case, the mKdV equation can be linearized. The growth factor  $\rho$  of the error in a typical mode of amplitude,

$$\delta_m^n = \rho^n e^{ijkh}, \quad (23)$$

where  $k$  is the mode number, and  $h$  is the element size. Substituting the Fourier mode (Eq. (23)) into Eq. (20) gives the growth factor,  $\rho$ , of the form:

$$\rho = \frac{a - ib}{a + ib}, \quad (24)$$

where:

$$\begin{aligned} a &= 302 \cos\left(\frac{kh}{2}\right) + 57 \cos\left(\frac{3kh}{2}\right) + \cos\left(\frac{5kh}{2}\right), \\ b &= [(120\varepsilon\lambda - 480\xi)\Delta t] \sin\left(\frac{kh}{2}\right) \\ &+ [(75\varepsilon\lambda + 60\xi)\Delta t] \sin\left(\frac{3kh}{2}\right) \\ &+ [(3\varepsilon\lambda + 60\xi)\Delta t] \sin\left(\frac{5kh}{2}\right). \end{aligned} \quad (25)$$

The modulus of  $|\rho|$  is 1; therefore, the linearized scheme is unconditionally stable.

### 4. Numerical examples and results

Numerical results of the mKdV equation are obtained for three problems: the motion of a single-solitary wave, interaction of two and three solitary waves. We use the error norm  $L_2$  as in the following:

$$L_2 = \|U^{\text{exact}} - U_N\|_2 \simeq \sqrt{h \sum_{j=1}^N |U_j^{\text{exact}} - (U_N)_j|^2}, \quad (26)$$

and the error norm  $L_\infty$  as in:

$$L_\infty = \|U^{\text{exact}} - U_N\|_\infty \simeq \max_j |U_j^{\text{exact}} - (U_N)_j|, \tag{27}$$

$$j = 1, 2, \dots, N - 1,$$

so that we can calculate the difference between analytical and numerical solutions at some specified times. Three of the many infinitely conserved quantities of mKdV Eq. (1) are:

$$I_1 = \int_a^b U dx \simeq h \sum_{j=1}^N U_j^n,$$

$$I_2 = \int_a^b U^2 dx \simeq h \sum_{j=1}^N (U_j^n)^2,$$

$$I_3 = \int_a^b \left[ U^4 - \frac{6\mu}{\varepsilon} (U_x)^2 \right] dx$$

$$\simeq h \sum_{j=1}^N \left[ (U_j^n)^4 - \frac{6\mu}{\varepsilon} (U_x)_j^n \right], \tag{28}$$

which correspond to conversation of mass, momentum, and energy, respectively [23,24]. In the simulation of solitary wave motion, the invariants  $I_1$ ,  $I_2$ , and  $I_3$  are monitored to check the conversation of the numerical algorithm.

**4.1. The motion of single solitary wave**

As the first problem, Eq. (1) is considered with respect to the boundary conditions  $U \rightarrow 0$  as  $x \rightarrow \pm\infty$  and the initial condition:

$$U(x, 0) = \sqrt{\frac{6c}{\varepsilon}} \operatorname{sech} \left[ \sqrt{\frac{c}{\mu}} (x - x_0) \right]. \tag{29}$$

The analytical solution of the mKdV can be written as:

$$U(x, t) = \sqrt{\frac{6c}{\varepsilon}} \operatorname{sech} \left[ \sqrt{\frac{c}{\mu}} (x - ct - x_0) \right], \tag{30}$$

where  $\varepsilon$ ,  $\mu$ ,  $c$ , and  $x_0$  are arbitrary constants. The conserved quantities of motion for a solitary wave of amplitude  $\sqrt{\frac{6c}{\varepsilon}}$  and width depending on  $\sqrt{\frac{c}{\mu}}$  may be evaluated analytically as in [13]:

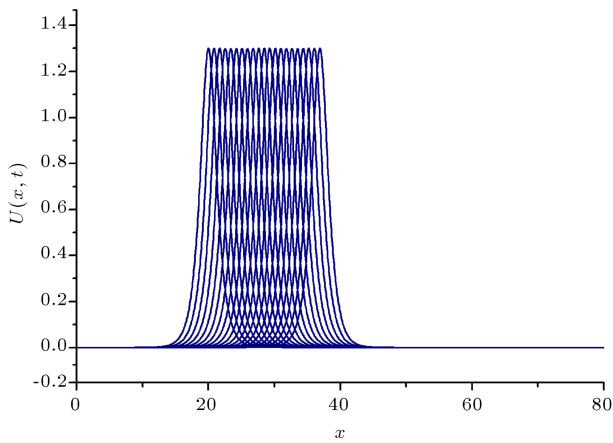
$$I_1 = \pi \sqrt{\frac{6\mu}{\varepsilon}}, \quad I_2 = \frac{12\sqrt{\mu c}}{\varepsilon},$$

$$I_3 = -\frac{64c^2}{\varepsilon^2} \sqrt{\frac{\mu}{c}}. \tag{31}$$

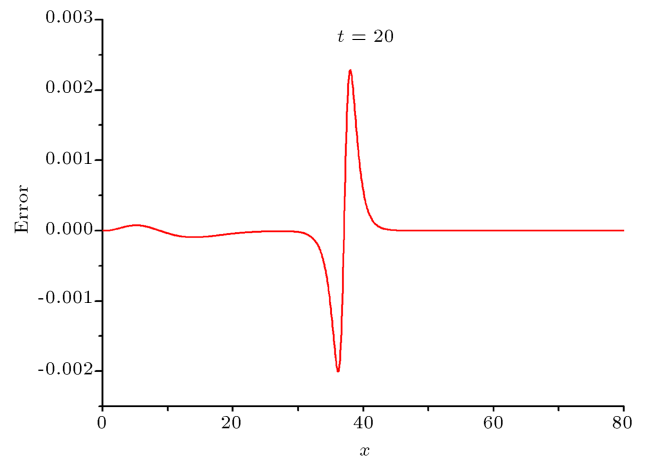
The values of the parameters  $\varepsilon = 3$ ,  $\mu = 1$ ,  $c = 0.845$ ,  $h = 0.1$ , and  $\Delta t = 0.01$  are chosen over the spatial interval  $[0, 80]$  to coincide with that of [13]. For these parameters, the solitary wave has an amplitude  $A = \sqrt{\frac{6c}{\varepsilon}} = 1.3$ . Numerical computations are done up to time  $t = 20$ . The obtained results are tabulated in Table 2. The conserved quantities and error norms  $L_2$  and  $L_\infty$  are shown at selected times. Table 2 represents a comparison of the values of the invariants and error norms obtained by the present method with those obtained by other method [13]. It is clearly seen from the table that the invariants remain almost constant during the computer run. It is observed from the table that the error norms obtained by our method are smaller than those given in [13]. In Figure 1, the numerical solutions are displayed at  $t = 0, 1, 2, \dots, 20$ . As seen from the figure, the soliton moves to the right at a constant speed and almost unchanged amplitude as time increases, as expected. In Figure 2, the motion of single soliton is plotted with  $\varepsilon = 3$ ,  $\mu = 1$ ,  $c = 0.845$ ,  $h = 0.1$ , and  $\Delta t = 0.01$  at selected times from  $t = 0$  to  $t = 20$ . Errors distributions at time  $t = 20$  are depicted for solitary waves amplitudes 1.3 in Figure 3 to show the errors between the analytical and numerical results over the problem domain.

**Table 2.** Invariants and error norms for single solitary wave with  $\varepsilon = 3$ ,  $\mu = 1$ ,  $c = 0.845$ ,  $h = 0.1$ , and  $\Delta t = 0.01$ ,  $0 \leq x \leq 80$ .

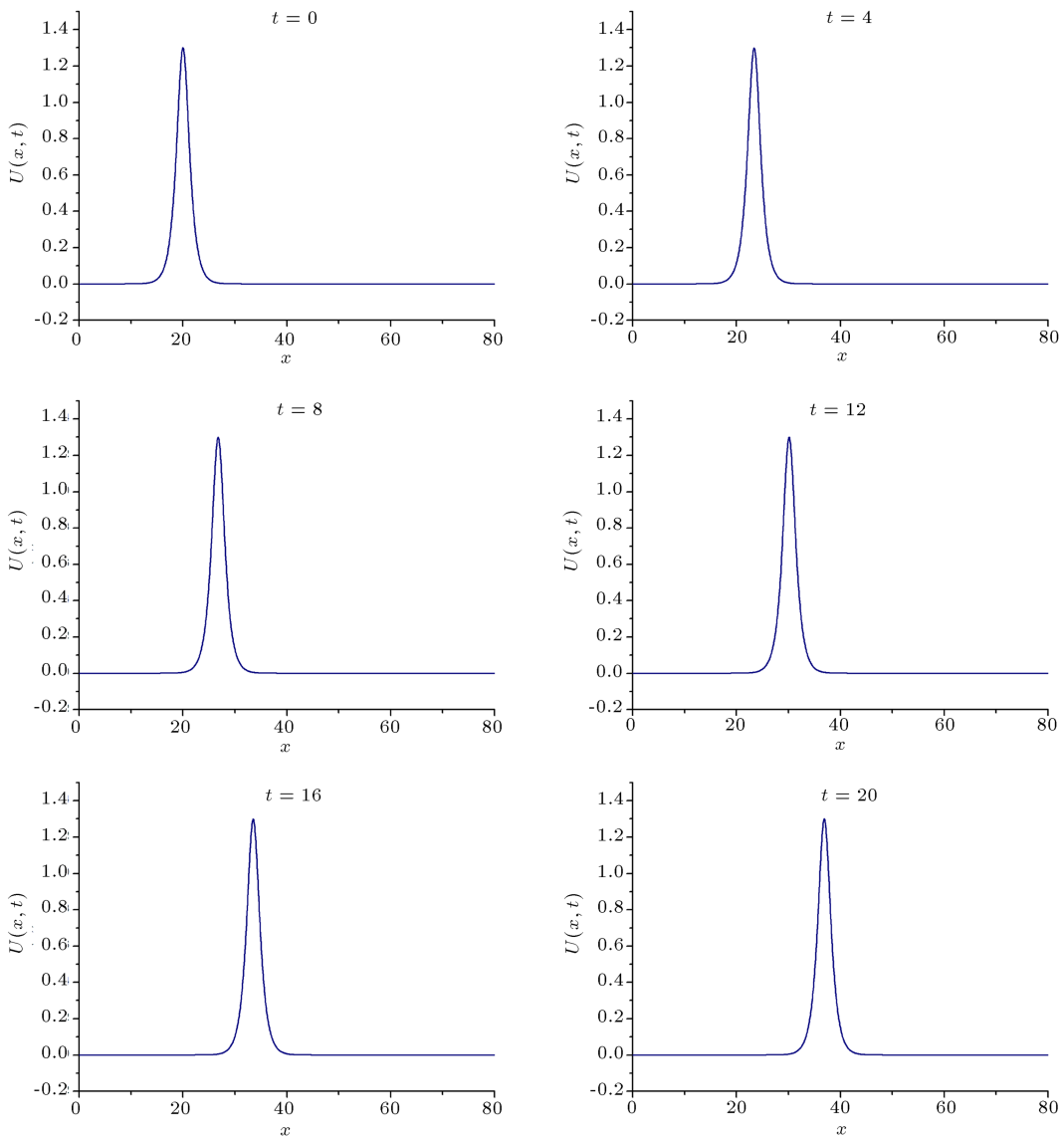
Results	t	1	5	10	15	20
$I_1$	Present	4.442866	4.442866	4.442866	4.442866	4.442866
	[13]	4.443000	4.443138	4.444142	4.443420	4.443171
$I_2$	Present	3.676941	3.676941	3.676941	3.676941	3.676941
	[13]	3.677069	3.677535	3.678094	3.678642	3.679192
$I_3$	Present	2.072795	2.073537	2.073699	2.073776	2.073846
	[13]	2.073575	2.074357	2.075303	2.076232	2.077161
$L_2$ -Error	Present	6.286951e-04	1.249516e-03	2.131860e-03	2.949376e-03	3.641638e-03
	[13]	—	—	—	—	—
$L_\infty$ -Error	Present	3.630992e-04	8.397466e-04	1.399503e-03	1.880855e-03	2.285638e-03
	[13]	1.206756e-03	3.621519e-03	5.942047e-03	7.626772e-03	8.642137e-03



**Figure 1.** Single solitary wave with  $\varepsilon = 3$ ,  $\mu = 1$ ,  $c = 0.845$ ,  $h = 0.1$ ,  $\Delta t = 0.010$ , and  $0 \leq x \leq 80$  at  $t = 0, 1, 2, \dots, 20$ .



**Figure 3.** Error with  $\varepsilon = 3$ ,  $\mu = 1$ ,  $c = 0.845$ ,  $h = 0.1$ , and  $\Delta t = 0.01$ ,  $0 \leq x \leq 80$ ,  $t = 20$ .



**Figure 2.** Single solitary wave with  $\varepsilon = 3$ ,  $\mu = 1$ ,  $c = 0.845$ ,  $h = 0.1$ ,  $\Delta t = 0.010$ , and  $0 \leq x \leq 80$  at  $t = 0, 4, 8, 12, 16$  and  $20$ .

**4.2. Interaction of two solitary waves**

As a second problem, we consider the interaction of two solitary waves by using the initial condition given by the linear sum of the two well-separated solitary waves having various amplitudes:

$$U(x, 0) = \sum_{i=1}^2 \alpha_i \operatorname{sech}[\beta_i(x - x_i)], \tag{32}$$

where  $\alpha_i = \sqrt{\frac{6c_i}{\varepsilon}}$ ,  $\beta_i = \sqrt{\frac{c_i}{\mu}}$ ,  $i = 1, 2$ , and  $c_i$  and  $x_i$  are arbitrary constants.

For the simulation, the parameters  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $c_1 = 2$ ,  $c_2 = 1$ ,  $x_1 = 15$ , and  $x_2 = 25$  are chosen over the range  $0 \leq x \leq 80$  to coincide with Ref. [13]. The experiment is run from  $t = 0$  to  $t = 20$  and the calculated values of the invariants  $I_1$ ,  $I_2$ , and  $I_3$  obtained by the present method with those obtained in [13] are compared in Table 3. It is seen that the obtained values of the invariants remain almost constant during the computer run.

Figure 4 shows the development of the interaction of two solitary waves. It is clear from the figure that at  $t = 0$ , the greater soliton is at the left position of the smaller soliton in the beginning of the run. With the increase in time, the greater soliton catches up the smaller one until at time  $t = 7$ , when the smaller soliton is absorbed. The overlapping

process continues until  $t = 8$ , while greater soliton has overtaken the smaller soliton and gotten in the process of the separating. At time  $t = 16$ , the interaction is complete, and the greater soliton has been separated completely.

**4.3. Interaction of three solitary waves**

As for the third problem, we study the behavior of the interaction of three solitary waves with different amplitudes and traveling in the same direction. So, we consider Eq. (1) with initial condition given by the linear sum of three well-separated solitary waves of different amplitudes:

$$U(x, 0) = \sum_{i=1}^3 \alpha_i \operatorname{sech}[\beta_i(x - x_i)], \tag{33}$$

where  $\alpha_j = \sqrt{\frac{6c_j}{\varepsilon}}$ ,  $\beta_j = \sqrt{\frac{c_j}{\mu}}$ ,  $i = 1, 2, 3$ ,  $c_i$ , and  $x_i$  are arbitrary constants.

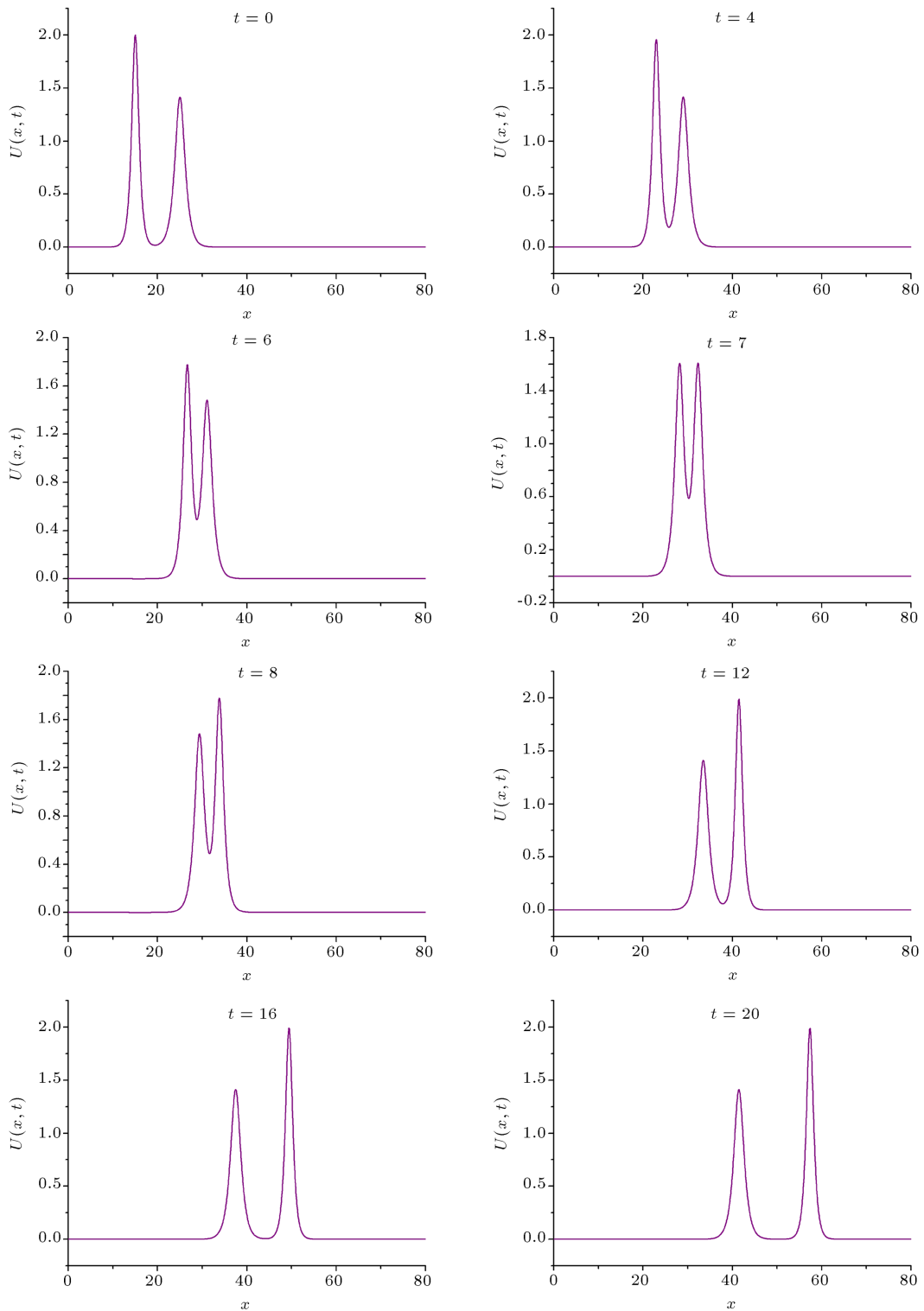
For the computational work, parameters  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $c_1 = 2$ ,  $c_2 = 1$ ,  $c_3 = 0.5$ ,  $x_1 = 15$ ,  $x_2 = 25$ , and  $x_3 = 35$  are taken over the interval  $0 \leq x \leq 80$ . Simulations are done up to time  $t = 20$ . Table 4 displays a comparison of the values of the invariants obtained by the present method with those obtained in [13]. It is seen from the table that the obtained values of the invariants remain

**Table 3.** Comparison of invariants for the interaction of two solitary waves with  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $c_1 = 2$ ,  $c_2 = 1$ ,  $x_1 = 15$ , and  $x_2 = 25$ ,  $0 \leq x \leq 80$ .

<i>I</i>	<i>t</i>	1	5	10	15	20
$I_1$	Present	8.885732	8.885732	8.885732	8.885732	8.885732
	[13]	8.886014	8.886776	8.889742	8.885983	8.884880
$I_2$	Present	9.659345	9.659345	9.659345	9.659345	9.659345
	[13]	9.659527	9.663714	9.662547	9.661071	9.661224
$I_3$	Present	10.270908	10.853235	10.954396	10.307195	10.338415
	[13]	10.239870	10.249000	10.246790	10.242580	10.242030

**Table 4.** Comparison of invariants for the interaction of three solitary waves with  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $c_1 = 2$ ,  $c_2 = 1$ ,  $c_3 = 0.5$ ,  $x_1 = 15$ ,  $x_2 = 25$ , and  $x_3 = 35$ ,  $0 \leq x \leq 80$ .

<i>I</i>	<i>t</i>	1	5	10	15	20
$I_1$	Present	13.328677	13.328677	13.328677	13.328677	13.328677
	[13]	13.329060	13.330630	13.338780	13.332640	13.332060
$I_2$	Present	12.519943	12.519943	12.519943	12.519943	12.519943
	[13]	12.520280	12.526260	12.540860	12.526660	12.524900
$I_3$	Present	11.321264	12.452085	12.476293	12.413843	11.499239
	[13]	11.249790	11.261270	11.288040	11.259970	11.256730



**Figure 4.** Interaction of two solitary waves with  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $c_1 = 2$ ,  $c_2 = 1$ ,  $x_1 = 15$ ,  $x_2 = 25$ , and  $0 \leq x \leq 80$  at  $t = 0, 4, 6, 7, 8, 12, 16$ , and  $20$ .

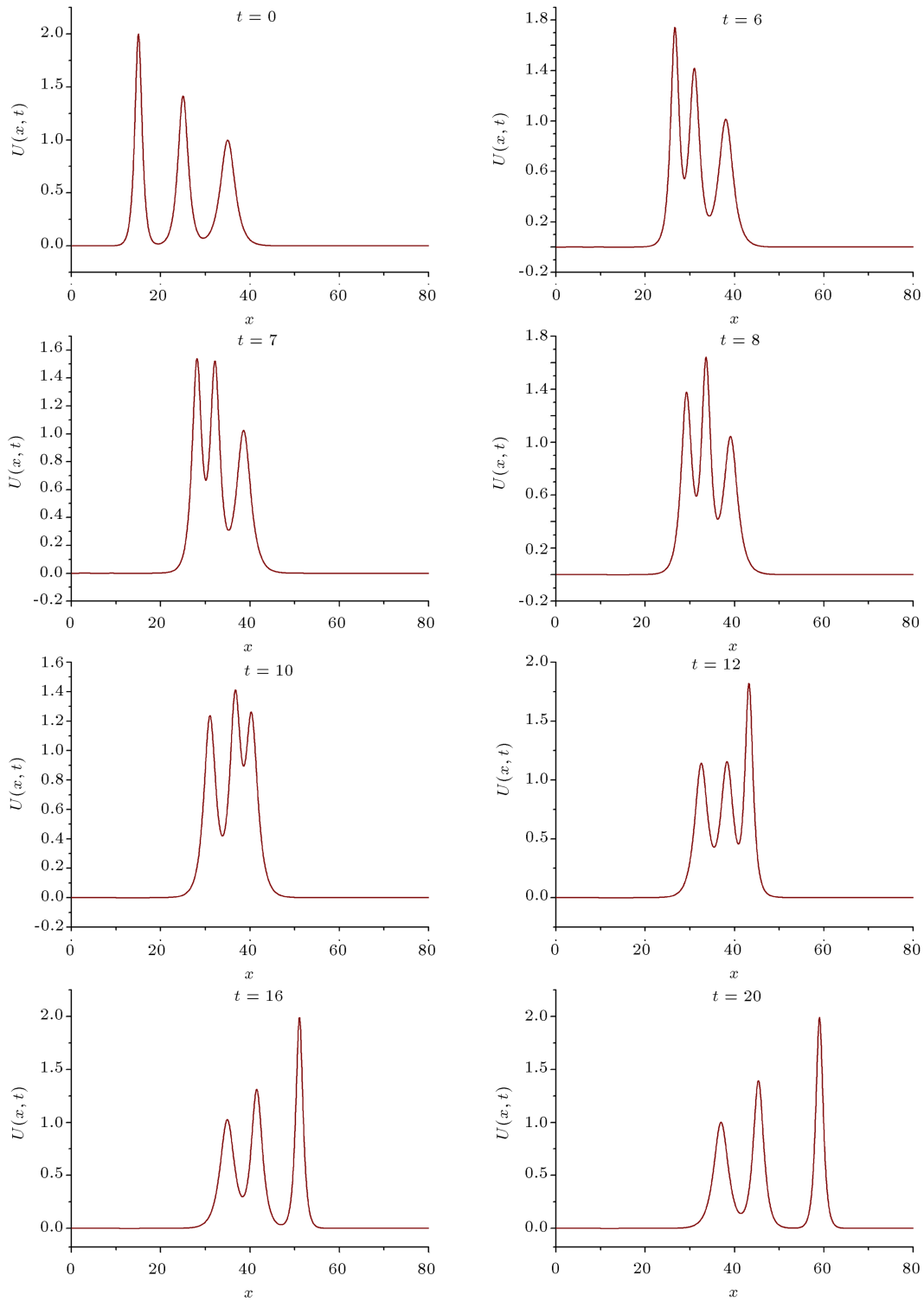
almost constant during the computer run. As seen from Figure 5, interaction started about time  $t = 6$ , overlapping processes occurred between time  $t = 6$  and  $t = 20$ , and waves started to resume their original shapes after the time  $t = 20$ .

### 5. Evolution of solitons

As for our last problem, evolution of the solitons has been studied using the Gaussian initial condition:

$$U(x, 0) = \exp(-x^2), \tag{34}$$





**Figure 5.** Interaction of three solitary waves with  $\varepsilon = 3$ ,  $\mu = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ ,  $c_1 = 2$ ,  $c_2 = 1$ ,  $c_3 = 0.5$ ,  $x_1 = 15$ ,  $x_2 = 25$ ,  $x_3 = 35$ , and  $0 \leq x \leq 80$  at  $t = 0, 6, 7, 8, 10, 12, 16$ , and  $20$ .

for various values of  $\mu$ . In this case, the behavior of the solution depends on the values of  $\mu$ . Therefore, the values of  $\mu = 0.1$  and  $\mu = 0.04$  are chosen at the region of the  $-50 \leq x \leq 50$ . The numerical computations are done up to  $t = 10$ . The values of the three invariants of motion for different  $\mu$  are presented in Table 5.

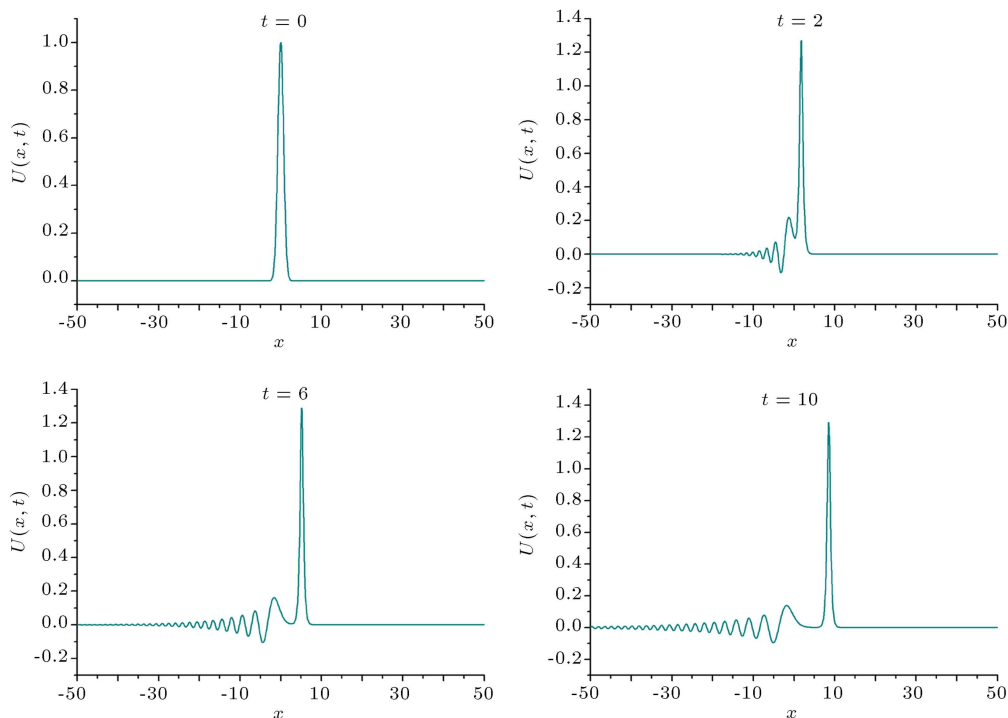
Also, Figures 6 and 7 illustrate the development of the Gaussian initial condition into solitary waves.

### 6. Conclusion

In this paper, a numerical treatment of the mKdV

**Table 5.** Invariants for Gaussian initial condition with  $\mu = 0.1$  and  $\mu = 0.04$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ , and  $-50 \leq x \leq 50$  at  $0 \leq t \leq 10$ .

$t$	$\mu = 0.1$			$\mu = 0.04$		
	$I_1$	$I_2$	$I_3$	$I_1$	$I_2$	$I_3$
0	1.772454	1.253314	.2355683	1.772454	1.253314	0.1958645
2	1.772454	1.253314	.2456447	1.772454	1.253314	0.1845277
4	1.772454	1.253314	.2099601	1.772454	1.253314	0.1912904
6	1.772454	1.253314	.2066328	1.772454	1.253314	0.2079808
8	1.772454	1.253314	.2010120	1.772454	1.253314	0.2229795
10	1.772454	1.253314	.1747070	1.772454	1.253314	0.2363841



**Figure 6.** Gaussian initial condition with  $\varepsilon = 3$ ,  $\mu = 0.1$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ , and  $-50 \leq x \leq 50$  at  $0 \leq t \leq 10$ .

equation has been introduced using cubic B-spline Petrov-Galerkin finite element method. To examine the performance of the scheme, four test problems have been studied. To show the performance of numerical scheme, the error norms  $L_2$  and  $L_\infty$  for single solitary wave and conserved quantities  $I_1$  and  $I_2$  for three test problems have been calculated. These calculations represent that the obtained error norms are smaller than the existing numerical results in the literature. The changes of the invariants are sufficiently small and the quantities of the invariants are consistent with those of [12]. Also, the linearized numerical scheme is unconditionally stable. Finally, the presented method can be reliably applied to obtain the numerical solution to the mKdV equation and similar types of non-linear problems.

The results of this paper stand on a very strong footing that is encouraged with a lot of future prospects in this avenue. Later, mKdV equation will be generalized to KdV equation with power law nonlinearity, so that KdV and mKdV equations will fall out as their special cases. The results of this generalized version will be disseminated elsewhere. In addition, potential KdV equation will also be addressed numerically using this scheme and other rich algorithms. Subsequently, the extension of KdV equation will be touched upon. These include Gardner’s equation that is otherwise known as KdV-mKdV equation.

In order to understand the double-layered shallow water wave dynamics, there are several proposed models. A couple of them are Bona-Chen equation as well as Gear-Grimshaw system. These vector-

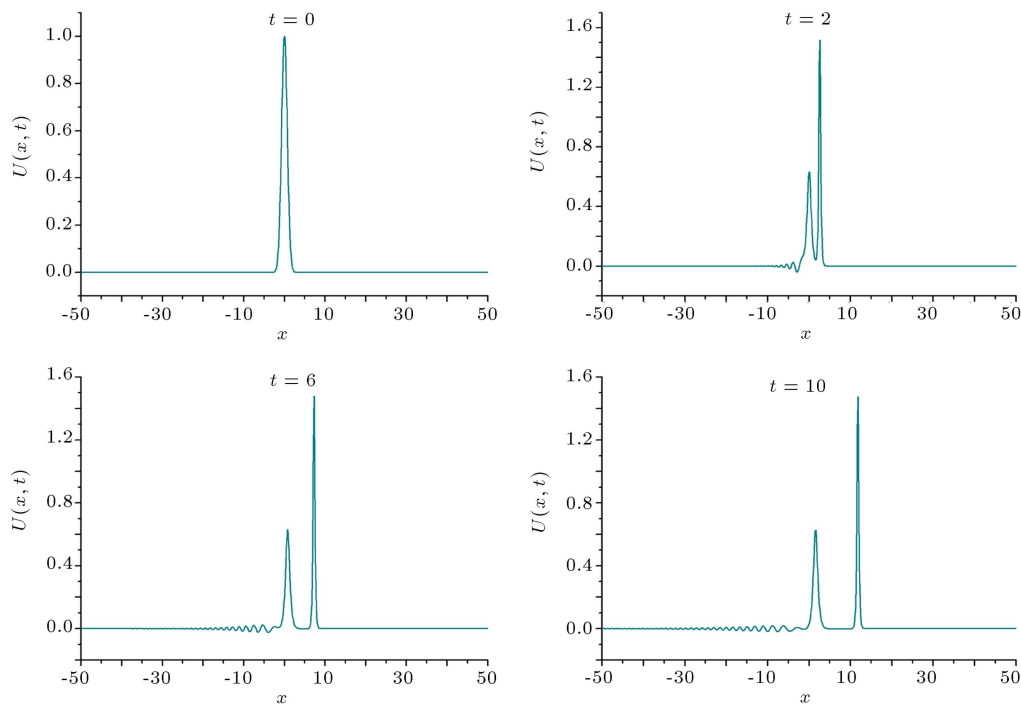


Figure 7. Gaussian initial condition with  $\varepsilon = 3$ ,  $\mu = 0.04$ ,  $h = 0.1$ ,  $\Delta t = 0.01$ , and  $-50 \leq x \leq 50$  at  $0 \leq t \leq 10$ .

coupled models will also be illustrated numerically to gain a better understanding of the dynamics of Exxon-Valdez oil spill in Alaska. The results of such research activities will be gradually and sequentially reported.

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