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Full Length Research Paper

Motion groups and circular helices in Lorentz 3-space

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In this paper, we find the curves which are orbits of points under the homothetic and helicoidal motion groups in Lorentz 3-space. Also, we show that if these curves are Frenet curves then their curvature and torsion are constant. So we can say that these curves are circular helix in Lorentz 3-space.

Key words: Helix, homothetic motion groups, helicoidal motion groups, Frenet curves.

INTRODUCTION

A helix in Euclidean 3-space E^3 is a parametrized curve whose tangent vector makes a constant angle with a fixed direction. In such case, this condition is equivalent to the ratio κ/τ is constant, where κ and τ denote the curvature and torsion of the curve, respectively, and $\tau \neq 0$. This situation was expressed by Lancret in 1802 and first proved by de Saint Venant in 1845. Particularly if both κ and τ are non-zero constant, the curve is called a circular helix [or W-curves by Struik (1988)]. There are a lot of interesting examples of helices in science and nature like DNA double, collagen triple helix and helical structures in fractal geometry. On the other hand, in E^3 the orbit of a point under a group of helicoidal motion is a curve with κ and τ are constant.

One could expect a similar result for the orbit of a point under a group of helicoidal motions in Lorentz 3-space E_1^3. First problem is in the definition, since the notion of angle is possible only for timelike and spacelike vectors. So we use another approximation which considers the curvature and torsion of a curve. For this, we have $\gamma : I \subset \mathbb{R} \to E_1^3$ a regular curve, orbit of a point under a group of helicoidal motion, parametrized by the arc length and search for the ones which are circular helix in E_1^3.

Consider the Lorentz 3-space E_1^3, that is, three dimensional real vector space $\mathbb{R}^3$ endowed with the metric $\langle , \rangle$ given by;

$$\langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle = x_1x_2 + y_1y_2 - z_1z_2$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$.

Definition 1. In three-dimensional Lorentz space,one-parameter homothetic motion of a body is generated by the transformation;

$$\begin{pmatrix} Y \\ 1 \end{pmatrix} \to \begin{pmatrix} \lambda A & u' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

where $A \in SO_1(3), A^t = \varepsilon A^{-1} \varepsilon$ and the signature matrix ε is the diagonal matrix $(\delta_{ij}\varepsilon_j)$ whose diagonal entries are $\varepsilon_1 = \varepsilon_2 = 1, \varepsilon_3 = -1.$ X and Y are real matrices of 3x1 type and λ is a homothetic scale. A, λ and u' are differentiable functions of C^∞ class of parameter t (Basdas, 1997; O'Neill, 1983).

We can express the 1-parameter homothetic motion groups in E_1^3 as a rotation around an axis,together with a translation in the direction of the axis or another axis. There are three types of homothetic motion groups depending on the character of the axis:

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1) If the axis L is timelike then we have a homothetic motion group expressed as;

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \lambda(t)\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} + h\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$$

2) If the axis L is spacelike then we have a homothetic motion group expressed as;

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \lambda(t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} + h\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

3) If the axis L is lightlike then we have a homothetic motion group expressed as;

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \lambda(t)\begin{pmatrix} 1-\frac{t^2}{2} & t & \frac{t^2}{2} \\ -t & 1 & t \\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} + h\begin{pmatrix} \frac{t^3}{3}-t \\ t^2 \\ \frac{t^3}{3}+t \end{pmatrix}$$

Here, $\lambda(t)$ is a differentiable function and h is a real scalar.

Particularly if we take $\lambda(t) = 1$ in the homothetic motion groups then we get helicoidal motion groups.

ORBIT OF A POINT UNDER A HOMOTETHIC MOTION GROUP IN E_1^3

Proposition 1. Let $p=(a,b,c)$ be a point in E_1^3 and α be a curve which is the orbit of p under a homothetic motion group in E_1^3. Then the character of α depends on t.

(1) If the axis is timelike then the orbit of p is;

$$\alpha(t) = (a\lambda(t)\cos t - b\lambda(t)\sin t, a\lambda(t)\sin t + b\lambda(t)\cos t, c\lambda(t)+ht)$$

and;

$$\alpha'(t) = (\lambda'(t)(a\cos t - b\sin t) -$$
$$\lambda(t)(a\sin t + b\cos t), \lambda'(t)(a\sin t + b\cos t) + \lambda(t)(a\cos t - b\sin t), c\lambda'(t)+h)$$

$$\langle \alpha'(t), \alpha'(t)\rangle = (a^2+b^2-c^2)\lambda'(t)^2 + (a^2+b^2)\lambda(t)^2 - 2ch\lambda'(t) - h^2$$

As we can see the character of the orbit is not independent from t.

But if we choose $c=0, p$ is in the plane that orthogonal to the axis, and $\lambda(t) = \frac{h}{\sqrt{a^2+b^2}}\cos t$ or, $\lambda(t) = \frac{h}{\sqrt{a^2+b^2}}\sin t$,then α will be a null curve. Here $a^2+b^2 \neq 0$

(2) If the axis is spacelike then the orbit of p is;

$$\alpha(t) = (a\lambda(t)+ht, b\lambda(t)\cosh t + c\lambda(t)\sinh t, b\lambda(t)\sinh t + c\lambda(t)\cosh t)$$

and;

$$\alpha'(t) = (a\lambda'(t)+h, \lambda'(t)(b\cosh t + c\sinh t) + \lambda(t)(b\sinh t + c\cosh t),$$

$$\lambda'(t)(b\sinh t + c\cosh t) + \lambda(t)(b\cosh t + c\sinh t))$$

$$\langle \alpha'(t), \alpha'(t)\rangle = (a^2+b^2-c^2)\lambda'(t)^2 - (b^2-c^2)\lambda(t)^2 + 2ah\lambda'(t) + h^2$$

As we can see the character of the orbit is not independent from t.

But if we choose $a=0, p$ is in the plane that is orthogonal to the axis, and $\lambda(t) = \frac{h}{\sqrt{b^2-c^2}}\cosh t,$ then α will be a null curve. Here $b^2-c^2 \neq 0$

(3) If the axis is lightlike then the orbit of p is

$$\alpha(t) = (\lambda(t)(a(1-\frac{t^2}{2}) + bt + c\frac{t^2}{2}) + h(\frac{t^3}{3}-t), \lambda(t)(-at+b+ct)+ht^2, \lambda(t)(-a\frac{t^2}{2}+bt+c(1+\frac{t^2}{2})) + h(\frac{t^3}{3}+t))$$

and;

$$\alpha'(t) = (\lambda'(t)(a - \frac{at^2}{2} + bt + \frac{ct^2}{2}) + \lambda(t)((c-a)t+b) + h(t^2-1), \lambda'(t)(-at+b+ct) + \lambda(t)(c-a) + 2ht,$$

$$\lambda'(t)(-\frac{at^2}{2} + bt + c + \frac{ct^2}{2}) + \lambda(t)((c-a)t+b) + h(t^2+1))$$

The character of the orbit is not independent from t. But if we choose $p=(0,b,0)$ and $\lambda(t) = \frac{h}{b}t^2,$ then α will be a null curve.

THE ORBIT OF A POINT UNDER HELICOIDAL MOTION GROUPS

Proposition 2. If we take $\lambda(t)=1$ in the homothetical motion groups then we have helicoidal motion groups.

(1) If the axis L is timelike then we have a helicoidal motion group expressed as;

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} + h\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$$

and if we take a point $p=(a,b,c)$ in E_1^3 then the orbit of p under this helicoidal motion is;

$$\alpha(t)=(a\cos t - b\sin t, a\sin t + b\cos t, c+ht)$$

and;

$$\left\langle \alpha'(t), \alpha'(t) \right\rangle = a^2+b^2-h^2$$

So the character of α is independent from t.

Corollary 1. Let α be a curve, which is the orbit of a point under the helicoidal motion group around timelike axis in E_1^3. In that case $\left\langle \alpha'(t), \alpha'(t) \right\rangle = a^2+b^2-h^2$ and if $a^2+b^2=h^2$ then α is a null curve, if $a^2+b^2>h^2$ then α is a spacelike curve and if $a^2+b^2<h^2$ then α is a timelike curve.

2) If the axis L is spacelike then we have a helicoidal motion group expressed as;

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} + h\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

and if we take a point $p=(a,b,c)$ in E_1^3 then the orbit of p under this helicoidal motion is;

$$\alpha(t)=(a+ht, b\cosh t + c\sinh t, b\sinh t + c\cosh t)$$

and;

$$\left\langle \alpha'(t), \alpha'(t) \right\rangle = -b^2+c^2+h^2$$

So the character of α is independent from t.

Corollary 2. Let α be a curve, which is the orbit of a point under the helicoidal motion group around spacelike axis in E_1^3. In that case $\left\langle \alpha'(t), \alpha'(t) \right\rangle = -b^2+c^2+h^2$ and if $b^2-c^2=h^2$ then α is a null curve,if $b^2-c^2<h^2$ then α is a spacelike curve and if $b^2-c^2>h^2$ then α is a timelike curve.

(3) If the axis L is lightlike then we have a helicoidal motion group expressed as;

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} 1-\frac{t^2}{2} & t & \frac{t^2}{2} \\ -t & 1 & t \\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} + h\begin{pmatrix} \frac{t^3}{3}-t \\ t^2 \\ \frac{t^3}{3}+t \end{pmatrix}$$

and then the orbit of p is;

$$\alpha(t)=(a+(b-h)t+\frac{c-a}{2}t^2+h\frac{t^3}{3}, b+(c-a)t+ht^2, c+(b+h)t+\frac{c-a}{2}t^2+h\frac{t^3}{3})$$

and;

$$\left\langle \alpha'(t), \alpha'(t) \right\rangle = (a-c)^2-4bh$$

So the character of α is independent from t.

Corollary 3. Let α be a curve, which is the orbit of a point under the helicoidal motion group around lightlike axis in E_1^3. In that case $\left\langle \alpha'(t), \alpha'(t) \right\rangle = (a-c)^2-4bh$ and if $(a-c)^2=4bh$ then α is a null curve, if $(a-c)^2>4bh$ then α is a spacelike curve and if $(a-c)^2<4bh$ then α is a timelike curve.

THE FRENET FRAME İN LORENTZ 3-SPACE

Let $\gamma: I \subset \mathbb{R} \to E_1^3$ be a nondegenerate curve parametrized by the arc-length and $\left\langle \gamma'(s), \gamma'(s) \right\rangle = \varepsilon.$ If γ is spacelike $\varepsilon=1,$ if γ is timelike $\varepsilon=-1.$

As $\gamma''(s)$ is orthogonal to $\gamma'(s),$ if γ is timelike then $\gamma''(s)$ is spacelike. But if γ is spacelike then $\gamma''(s)$ can be spacelike,timelike or lightlike. In the case that $\gamma''(s)$ is not lightlike and $\gamma''(s) \neq 0$ for any $s \in I$ then γ is called Frenet curve.

As $T(s) = \gamma'(s)$ is the tangent vector then the curvature of γ is defined as;

$$\kappa(s) = \left\| \gamma''(s) \right\|$$

the normal vector is defined as;

$$N(s) = \frac{\gamma''(s)}{\left\| \gamma''(s) \right\|}$$

and finally the binormal vector defined as;

$$B(s) = T(s) \text{x} N(s)$$

Let $\left\langle N(s), N(s) \right\rangle = \delta$. If $N(s)$ is spacelike then $\delta = 1$ and if $N(s)$ is timelike then $\delta = -1$. Also $\left\langle B(s), B(s) \right\rangle = -\varepsilon\delta$. $\{T(s), N(s), B(s)\}$ is called the Frenet frame and the derivatives of these three vectors are called Frenet equations which are;

$$T'(s) = \kappa(s) N(s)$$

$$N'(s) = -\varepsilon\delta\kappa(s) T(s) + \tau B(s)$$

$$B'(s) = \varepsilon\tau N(s)$$

Here τ is the torsion of γ and from the second Frenet equation we have $\tau(s) = -\varepsilon\delta \left\langle N'(s), B(s) \right\rangle$ (Lopez, 2008).

Theorem 1. If the orbit of a point under a helicoidal motion group is a Frenet curve in E_1^3 then the curvature and the torsion of this curve are constant.

Proof 1. If L is timelike.

Let $p = (a,b,c) \notin L$ be a point in E_1^3. The orbit of p under the helicoidal motion group is

$$\alpha(t) = (a\cos t - b\sin t, a\sin t + b\cos t, c + ht)$$

and if we take arc-length parametrized curve then we have;

$$\beta(s) = (a\cos\frac{s}{k} - b\sin\frac{s}{k}, a\sin\frac{s}{k} + b\cos\frac{s}{k}, c + h\frac{s}{k})$$

and;

$$T(s) = \frac{1}{k}\left(-a\sin\frac{s}{k} - b\cos\frac{s}{k}, a\cos\frac{s}{k} - b\sin\frac{s}{k}, h \right)$$

$$N(s) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\cos\frac{s}{k} + b\sin\frac{s}{k}, -a\sin\frac{s}{k} - b\cos\frac{s}{k}, 0 \right)$$

$$B(s) = \frac{1}{k\sqrt{a^2+b^2}}(h(a\sin\frac{s}{k} + b\cos\frac{s}{k}), h(a\cos\frac{s}{k} - b\sin\frac{s}{k}), -(a^2+b^2))$$

here $k = \sqrt{\varepsilon(a^2+b^2-h^2)}$ and with an easy computation we have

$$\kappa(s) = \varepsilon\frac{\sqrt{(a^2+b^2)}}{a^2+b^2-h^2}$$

and;

$$\tau(s) = -\frac{h}{a^2+b^2-h^2}$$

Here $\left\langle \alpha'(t), \alpha'(t) \right\rangle = a^2+b^2-h^2 \neq 0$ and $a^2+b^2 \neq 0$

Proof 2. If L is spacelike.

Let $p = (a,b,c) \notin L$ be a point in E_1^3. The orbit of p under the helicoidal motion group is;

$$\alpha(t) = (a + ht, b\cosh t + c\sinh t, b\sinh t + c\cosh t)$$

and if we take arc-length parametrized curve then we have;

$$\beta(s) = (a + h\frac{s}{k}, b\cosh\frac{s}{k} + c\sinh\frac{s}{k}, b\sinh\frac{s}{k} + c\cosh\frac{s}{k})$$

and;

$$T(s) = \frac{1}{k}\left(h, b\sinh\frac{s}{k} + c\cosh\frac{s}{k}, b\cosh\frac{s}{k} + c\sinh\frac{s}{k} \right)$$

$$N(s)=\frac{1}{\sqrt{b^2-c^2}}\left(0,b\cosh\frac{s}{k}+c\sinh\frac{s}{k},b\sinh\frac{s}{k}+c\cosh\frac{s}{k}\right)$$

$$B(s)=\frac{1}{k\sqrt{b^2-c^2}}(c^2-b^2,$$
$$h(b\sinh\frac{s}{k}+c\cosh\frac{s}{k}),-h(b\cosh\frac{s}{k}+c\sinh\frac{s}{k}))$$

here $k=\sqrt{\varepsilon(-b^2+c^2+h^2)}$ and with an easy computation we have;

$$\kappa(s)=\frac{\sqrt{b^2-c^2}}{\varepsilon(-b^2+c^2+h^2)}$$

and;

$$\tau(s)=-\frac{h}{\varepsilon(-b^2+c^2+h^2)}$$

Here $\left\langle\alpha'(t),\alpha'(t)\right\rangle=-b^2+c^2+h^2\neq 0$ and $b^2-c^2>0$

Proof. If L is lightlike.

Let $p=(a,b,c)\notin L$ be a point in E_1^3 . The orbit of p under the helicoidal motion group is;

$$\alpha(t)=(a+(b-h)t+\frac{c-a}{2}t^2+h\frac{t^3}{3},b+(c-a)t+ht^2,c+(b+h)t+\frac{c-a}{2}t^2+h\frac{t^3}{3})$$

and if we take arc-length parametrized curve then we have;

$$\beta(s)=(\frac{h}{3k^3}s^3+\frac{c-a}{2k^2}s^2+\frac{b-h}{k}s+a,\frac{h}{k^2}s^2+\frac{c-a}{k}s+b,\frac{h}{3k^3}s^3+\frac{c-a}{2k^2}s^2+\frac{b+h}{k}s+c)$$

here $k=\sqrt{(c-a)^2-4bh}$ and;

$$T(s)=(\frac{hs^2}{k^3}+\frac{(c-a)s}{k^2}+\frac{(b-h)}{k},\frac{2hs}{k^2}+\frac{c-a}{k},\frac{hs^2}{k^3}+\frac{(c-a)s}{k^2}+\frac{(b+h)}{k})$$

$$N(s)=(\frac{s}{k}+\frac{c-a}{2h},1,\frac{s}{k}+\frac{c-a}{2h})$$

$$B(s)=(\frac{hs^2}{k^3}+\frac{(c-a)s}{k^2}+\frac{(c-a)^2}{2hk}-\frac{(b+h)}{k},\frac{2hs}{k^2}+\frac{c-a}{hk},\frac{hs^2}{k^3}+\frac{(c-a)s}{k^2}+\frac{(c-a)^2}{2hk}-\frac{(b-h)}{k})$$

and with an easy computation we have;

$$\kappa=\frac{2h}{k^2}$$

and;

$$\tau=\frac{2h}{k^2}$$

Here $\left\langle\alpha'(t),\alpha'(t)\right\rangle=(c-a)^2-4bh\neq 0$.

As we can see in all three cases κ and τ are constant. Thus we can give the following theorem.

Theorem 2. If the orbit of a point under a helicoidal motion group in E_1^3 is a Frenet curve then this curve is a circular helix in E_1^3.

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