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Helicoidal surfaces in Minkowski space with constant mean curvature and constant Gauss curvature

Research Article

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Abstract: We classify all helicoidal non-degenerate surfaces in Minkowski space with constant mean curvature whose generating curve is a the graph of a polynomial or a Lorentzian circle. In the first case, we prove that the degree of the polynomial is 0 or 1 and that the surface is ruled. If the generating curve is a Lorentzian circle, we prove that the only possibility is that the axis is spacelike and the center of the circle lies on the axis.

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1. Introduction and statement of results

Consider the Lorentz–Minkowski space $\mathbb{E}_1^3$, that is, the three-dimensional real vector space $\mathbb{R}^3$ endowed with the metric $\langle \cdot, \cdot \rangle$ given by

$$\langle (x, y, z), (x', y', z') \rangle = xx' + yy' - zz',$$

where (x, y, z) are the canonical coordinates of $\mathbb{R}^3$. A helicoidal motion of $\mathbb{E}_1^3$ is a Lorentzian rotation around an axis L followed by a translation. A *helicoidal surface* in Minkowski space $\mathbb{E}_1^3$ is a surface invariant under a uniparametric group $G_{L,h} = \{\phi_t : t \in \mathbb{R}\}$ of helicoidal motions. Each group of helicoidal motions is characterized by an axis L and a pitch $h \neq 0$, and each helicoidal surface is determined by a group of helicoidal motions and a generating curve γ, i.e. a helicoidal surface can be parametrized as $X(s, t) = \phi_t(\gamma(s))$, $t \in \mathbb{R}$, $s \in I \subset \mathbb{R}$. Helicoidal surfaces in $\mathbb{E}_1^3$ with prescribed curvature were considered in [1], and in the case when the axis is lightlike in [6, 7] (see also [15]). In the limit case $h = 0$, the surface is rotational and the mean curvature equation is an ordinary differential equation of second order that has a first integral.

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The first part of this paper is motivated by the results from [2], where authors studied helicoidal surfaces generated by a straight-line, called *helicoidal ruled surfaces*. A particular case of such surfaces are the surfaces called *right Lorentzian helicoids* considered by Dillen and Kühnel, which appear when the axis L is timelike or spacelike and the curve γ is one of the coordinate axes of $\mathbb{R}^3$. There are three types of such surfaces: the helicoid of first kind if L is timelike, the helicoid of second type if L is spacelike and γ is the y-axis, and the helicoid of third type if L is spacelike and γ is the z-axis. All these three surfaces have zero mean curvature. When the axis L is lightlike, there are two known helicoidal surfaces generated by a straight-line called in the literature the Lie minimal surface (or the Cayley surface) and the parabolic null cylinder [2, 13, 16]. Both surfaces also have zero mean curvature. In this paper we consider a generalization of this setting with γ being the graph of a polynomal $f(s)=\sum_{n=0}^{m}a_ns^n$ instead of a straight line. Furthermore, we suppose that the mean curvature is constant, not necessarily zero. We look for conditions under which the corresponding helicoidal surface has constant mean curvature. We establish the following result.

Theorem 1.1.

Consider a helicoidal non-degenerate surface in $\mathbf{E}_1^3$ with constant mean curvature H whose generating curve is the graph of a polynomial $f(s)=\sum_{n=0}^{m}a_ns^n$. Then $m\leq 1$, that is, the generating curve is a straight-line. Moreover, and after a rigid motion of $\mathbf{E}_1^3$:

1. *If the axis is timelike $L=\langle(0,0,1)\rangle$, the surface is either the helicoid of first kind, $H=0$, or the surface $X(s,t)=(s\cos t, s\sin t, \pm s+a_0+ht)$, $a_0\in\mathbb{R}$, with $H=1/h$, or the Lorentzian cylinder of equation $x^2+y^2=r^2$ whose mean curvature is $H=1/(2r)$.*
2. *If the axis is spacelike $L=\langle(1,0,0)\rangle$, then $H=0$. The surface is either the helicoid of second kind, or the helicoid of third kind, or the surface parametrized by $X(s,t)=\big(ht,(\pm s+a_0)\sinh t+s\cosh t,(\pm s+a_0)\cosh t+s\sinh t\big)$, $a_0\neq 0$.*
3. *If the axis is lightlike $L=\langle(1,0,1)\rangle$, then $H=0$ and the surface is either the Cayley surface or the parabolic null cylinder.*

In Section 4, we will also study helicoidal surfaces with $H^2-K=0$. Recall that in Minkowski space, there are non-umbilical timelike surfaces with $H^2-K=0$. We will find all such surfaces when the generating curve is the graph of a polynomial.

The motivation of the second part of this article has its origin in the helicoidal surfaces whose generating curve is a Lorentzian circle of $\mathbf{E}_1^3$, called in [13] *right circular cylinders*. Let us start with two examples. Consider the Lorentzian circle given by $\gamma(s)=(0,r\cosh s,r\sinh s)$, $r>0$, and apply a group of helicoidal motions whose axis is $L=\langle(1,0,0)\rangle$. The surface generated by γ is the timelike hyperbolic cylinder of equation $y^2-z^2=r^2$ and its mean curvature is constant with $H=1/(2r)$. Similarly, if one considers the curve $\gamma(s)=(0,r\sinh s,r\cosh s)$, the surface obtained under rotations of the above group is the spacelike hyperbolic cylinder of the equation $y^2-z^2=-r^2$, in this case $H=1/(2r)$ again. In this paper we consider a general problem of finding all helicoidal surfaces with constant mean curvature whose generating curve is a Lorentzian circle of $\mathbb{R}^3$. It turns out that the above examples describe all the possibilities.

Theorem 1.2.

Consider a helicoidal non-degenerate surface in $\mathbf{E}_1^3$ with constant mean curvature H whose generating curve is a Lorentzian circle of $\mathbf{E}_1^3$. Then the axis of the surface is spacelike and $H\neq 0$. Moreover, the center of the circle lies on the axis and, up to a rigid motion of $\mathbf{E}_1^3$, the surface is one of the hyperbolic cylinders of equations $y^2-z^2=\pm r^2$.

We finish this article by studying helicoidal surfaces with constant Gauss curvature K. When the axis is timelike, the Gauss curvature K of the second surface from Theorem 1.1 is $K=1/h^2$. On the other hand, all the surfaces from Theorem 1.2 have $K=0$. We prove that these are the only helicoidal surfaces with constant Gauss curvature.

Theorem 1.3.

Consider a helicoidal non-degenerate surface in $\mathbf{E}_1^3$ with constant Gauss curvature K.

1. *If the generating curve is the graph of a polynomial $f(s)=\sum_{n=0}^{m}a_ns^n$, then $m\leq 1$. If the axis is timelike, the surface is either the Lorentzian cylinder of equation $x^2+y^2=r^2$, $K=0$, or the surface $X(s,t)=(s\cos t,s\sin t,\pm s+a_0+ht)$ with $K=1/h^2$; if the axis is spacelike, the surface is $X(s,t)=\big(ht,(\pm s+a_0)\sinh t+s\cosh t,(\pm s+a_0)\cosh t+s\sinh t\big)$, $a_0\neq 0$ with $K=0$; if the axis is lightlike, the surface is the parabolic null cylinder with $K=0$.*

2. *If the generating curve is a circle, then the axis is spacelike, $K = 0$, the center of the circle lies on the axis and the surface is one of the hyperbolic cylinders of equations $y^2 - z^2 = \pm r^2$.*

Throughout this work, we will assume that a helicoidal motion is not a rotation, that is, $h \neq 0$. Rotational surfaces with constant mean curvature or constant Gauss curvature were studied in [3, 4, 8, 10, 11].

The article is organized as follows. In Section 2 we recall the parametrizations of helicoidal surfaces as well as the definition of a Lorentzian circle in $\mathbf{E}_1^3$. In Section 3 we recall the definition of the mean curvature and the Gauss curvature of a non-degenerate surface, describing the way to compute H and K in local coordinates. In Section 4 we prove Theorem 1.1, and in Sections 5 and 6 we prove, Theorems 1.2 and 1.3, respectively.

2. Description of helicoidal surfaces of E_1^3

In this section we describe the parametrization of a helicoidal surface in $\mathbf{E}_1^3$ and recall the notion of a Lorentzian circle. The metric $\langle \cdot, \cdot \rangle$ divides vectors of $\mathbf{E}_1^3$ into three types according to its causal character. A vector $v \in \mathbf{E}_1^3$ is called spacelike (resp. timelike, lightlike) if $\langle v, v\rangle > 0$ or $v = 0$ (resp. $\langle v, v\rangle < 0$, $\langle v, v\rangle = 0$ and $v \neq 0$). Given a vector subspace $U \subset \mathbf{E}_1^3$, we say that U is called spacelike (resp. timelike, lightlike) if the induced metric is positive definite (resp. non-degenerate of index 1, degenerated and $U \neq \{0\}$). Recall that there is the following classification of the Lorentzian motion groups.

Proposition 2.1.

A helicoidal Lorentzian motion group is a uniparametric group of Lorentzian rigid motions which are non-trivial. A group of helicoidal motions group $G_{L,h} = \{\phi_t : t \in \mathbb{R}\}$ is determined by an axis L and a pitch $h \in \mathbb{R}$. After a change of coordinates any helicoidal motion group has one of the following parametrizations.

1. *If L is timelike, then $L = \langle (0, 0, 1)\rangle$ and*

$$\phi_t(a, b, c) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + h \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}.$$

2. *If L is spacelike, then $L = \langle (1, 0, 0)\rangle$ and*

$$\phi_t(a, b, c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + h \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}.$$

3. *If L is lightlike, then $L\langle (1, 0, 1)\rangle$ and*

$$\phi_t(a, b, c) = \begin{pmatrix} 1 - t^2/2 & t & t^2/2 \\ -t & 1 & t \\ -t^2/2 & t & 1 + t^2/2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + h \begin{pmatrix} t^3/3 - t \\ t^2 \\ t^3/3 + t \end{pmatrix}.$$

If $h = 0$, then we obtain a rotation group around the axis L.

If the axis is spacelike or timelike, the translation vector is the direction of the axis. The following result is obtained in [13, Lemma 2.1] and it says how to parametrize a helicoidal surface.

Proposition 2.2.

Let S be a surface in $\mathbf{E}_1^3$ invariant under a group of helicoidal motions $G_{L,h} = \{\phi_t : t \in \mathbb{R}\}$. Then there exists a planar curve $\gamma = \gamma(s)$, $s \in I$, such that $S = \{\phi_t(\gamma(s)) : s \in I, t \in \mathbb{R}\}$ (the curve γ is called a generating curve of S). Moreover,

1. *if L is timelike, γ lies in a plane containing L;*
2. *if L is spacelike, γ lies in a plane orthogonal to L;*
3. *if L is lightlike, γ lies in the only degenerate plane containing L.*

Thus, by Propositions 2.1 and 2.2, a helicoidal surface in $\mathbf{E}_1^3$ locally parametrizes as follows.

1. If the axis is timelike, with $L = \langle(0,0,1)\rangle$ and $\gamma(s) = (s, 0, f(s))$, then

$$X(s,t) = (s\cos t, s\sin t, ht + f(s)), \qquad s \in I, \quad t \in \mathbb{R}. \tag{1}$$

2. If the axis is spacelike, with $L = \langle(1,0,0)\rangle$ and $\gamma(s) = (0, s, f(s))$, then

$$X(s,t) = \big(ht, s\cosh t + f(s)\sinh t, s\sinh t + f(s)\cosh t\big), \qquad s \in I, \quad t \in \mathbb{R}. \tag{2}$$

3. If the axis is lightlike, with $L = \langle(1,0,1)\rangle$ and $\gamma(s) = (f(s), s, f(s))$, then

$$X(s,t) = \left(st + h\left(\frac{t^3}{3} - t\right) + f(s),\ s + ht^2,\ st + h\left(\frac{t^3}{3} + t\right) + f(s)\right), \qquad s \in I, \quad t \in \mathbb{R}. \tag{3}$$

Remark 2.3.

In [2] the authors define a ruled helicoidal surface as a helicoidal surface generated by a straight-line. Any ruled helicoidal surface is both a ruled surface as a helicoidal surface. However, there are ruled surfaces that are helicoidal surfaces but are not generated by a straight-line in the sense of Proposition 2.2. For example, the timelike hyperbolic cylinder of equation $y^2 - z^2 = r^2$ is helicoidal whose axis is $L = \langle(1,0,0)\rangle$, and it is also a ruled surface, but the intersection of the surface with the plane $x = 0$ is the curve γ parametrized by $\gamma(s) = (0, r\cosh s, r\sinh s)$, $s \in \mathbb{R}$, which is not a straight-line. In fact, this surface is invariant under of all helicoidal motions with axis L and arbitrary pitch h [13]. On the other hand, this surface can be viewed as a surface of revolution with axis L obtained by rotating the curve $\alpha(s) = (s, 0, r)$. This curve α is not a generating curve according to Proposition 2.2.

Finally we recall the definition of a Lorentzian circle in Minkowski space $\mathbf{E}_1^3$ (see [9]).

Definition 2.4.

A Lorentzian circle in $\mathbf{E}_1^3$ is the orbit of a point under a group of rotations.

Let $p = (a, b, c)$ be a point of $\mathbf{E}_1^3$ and let $G_L = \{\phi_t : t \in \mathbb{R}\}$ a group of rotations with axis L. Let us describe the trajectory of p under G_L, that is, $\alpha(t) = \phi_t(p)$, $t \in \mathbb{R}$. We assume that $p \notin L$ as otherwise α reduces in one point. Depending on the causal character of L, there are three types of circles.

1. The axis is timelike, $L = \langle(0,0,1)\rangle$. Then $\alpha(t) = (a\cos t - b\sin t, b\cos t + a\sin t, c)$. This curve is an Euclidean circle of radius $\sqrt{a^2 + b^2}$ contained in the plane $z = c$.
2. The axis is spacelike, $L = \langle(1,0,0)\rangle$. Then $\alpha(t) = (a, b\cosh t + c\sinh t, c\cosh t + b\sinh t)$ with $|\alpha'(t)|^2 = -b^2 + c^2$. Depending on the relation between b and c, we distinguish the following three sub-cases.
   - If $b^2 < c^2$, then α is spacelike and it meets the z-axis at one point. After a translation, we can assume that $p = (0,0,c)$. Then $\alpha(t) = (0, c\sinh t, c\cosh t)$. This curve is the hyperbola of equation $z^2 - y^2 = c^2$ in the plane $x = 0$.
   - If $b^2 = c^2$, then α is lightlike, $\alpha(t) = (a, \pm c(\cosh t + \sinh t), c(\cosh t + \sinh t))$. Thus α is one of the two straight-lines $y = \pm z$ in the plane $x = a$.
   - If $b^2 > c^2$, then α is timelike and it meets the y-axis at one point. We may suppose that $p = (0, b, 0)$, then $\alpha(t) = (0, b\cosh t, b\sinh t)$. This curve is the hyperbola of equation $y^2 - z^2 = b^2$ in the plane $x = 0$.
3. The axis is lightlike, $L = \langle(1,0,1)\rangle$ and $p = (a, 0, c)$. As $|\alpha'(t)|^2 = (a-c)^2$ and $p \notin L$, α is the spacelike curve $\alpha(t) = (a, 0, c) + (c-a)t(0,1,0) + (c-a)/2t^2(1,0,1)$. This curve lies in the plane $x - z = a - c$ and from the Euclidean viewpoint, it is a parabola with the axis parallel to $(1,0,1)$.

3. Curvature of a non-degenerate surface

This section partially is based on [12, 14, 17]. An immersion $x\colon M \to \mathbf{E}_1^3$ of a surface M is called spacelike (resp. timelike) if the tangent plane T_pM is spacelike (resp. timelike) for all $p \in M$. We also say that M is spacelike (resp. timelike). In both cases, we say that the surface is non-degenerate.

Let $\mathfrak{X}(M)$ be the class of tangent vector fields of M. Denote by ∇^0 the Levi-Civita connection of $\mathbf{E}_1^3$ and by ∇ the induced connection on M by the immersion x, that is, $\nabla_X Y = (\nabla^0_X Y)^\top$, where $\top$ denotes the tangent part of the vector field $\nabla^0_X Y$. We have the decomposition

$$\nabla^0_X Y = \nabla_X Y + \sigma(X,Y), \tag{4}$$

called the Gauss formula. Here $\sigma(X,Y)$ is the normal part of the vector $\nabla^0_X Y$. Now consider ξ a normal vector field to x and let $-\nabla^0_X\xi$. Denote by $A_\xi(X)$ its tangent component, that is, $A_\xi(X) = -(\nabla^0_X\xi)^\top$. From (4), we have

$$\langle A_\xi(X), Y\rangle = \langle \sigma(X,Y), \xi\rangle. \tag{5}$$

The map $A_\xi\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ is called the Weingarten endomorphism of ξ. As σ is symmetric, we obtain from (5) that $\langle A_\xi(X), Y\rangle = \langle X, A_\xi(Y)\rangle$. This means that A_ξ is self-adjoint with respect to the metric $\langle\cdot,\cdot\rangle$ of M. Since our results are local, we only need local orientability, which is trivially satisfied. However, we recall that a spacelike surface is globally orientable. Denote by N the Gauss map on M. Define ϵ by $\langle N,N\rangle = \epsilon$, where $\epsilon = -1$ (resp. 1) if the immersion is spacelike (resp. timelike). If we take $\xi = N$, then $\langle \nabla^0_X N, N\rangle = 0$, hence the normal part of $\nabla_X N$ vanishes, and we arrive to the Weingarten formula

$$-\nabla^0_X N = A_N(X). \tag{6}$$

Definition 3.1.
The Weingarten endomorphism at $p \in M$ is defined by $A_p\colon T_pM \to T_pM$, $A_p = A_{N(p)}$, that is, if $v \in T_pM$ and $X \in \mathfrak{X}(M)$ is a tangent vector field that extends v, then $A_p(v) = (A(X))_p$. Moreover, from (6), $A_p(v) = -(dN)_p(v)$, $v \in T_pM$, where $(dN)_p$ is the usual differentiation in $\mathbf{E}_1^3$ of the map N at p.

As $\sigma(X,Y)$ is proportional to N, the Gauss formula (4) and (5) give $\sigma(X,Y) = \epsilon\langle\sigma(X,Y),N\rangle N = \epsilon\langle A(X),Y\rangle N.$, so the Gauss formula takes the form $\nabla^0_X Y = \nabla_X Y + \epsilon\langle A(X),Y\rangle N$.

Definition 3.2.
Given a non-degenerate immersion, the mean curvature vector field $\overrightarrow{H}$ and the Gauss curvature K are defined as follows:

$$\overrightarrow{H} = \frac{1}{2}\,\mathrm{trace}_{\mathrm{I}}\sigma, \qquad K = \epsilon\,\frac{\det\sigma}{\det \mathrm{I}},$$

where the subscript I means that the computation is done with respect to the metric $\mathrm{I} = \langle\cdot,\cdot\rangle$. The mean curvature function H is given by $\overrightarrow{H} = HN$, that is, $H = \epsilon\langle\overrightarrow{H}, N\rangle$.

In terms of the Weingarten endomorphism A, the expressions of H and K have the form

$$H = \frac{\epsilon}{2}\,\mathrm{trace}\,A, \qquad K = \epsilon\det A.$$

In this work we will compute H and K using a parametrization of the surface. Let $X\colon U\subset\mathbb{R}^2 \to \mathbf{E}_1^3$ be a parametrization with $X = X(u,v)$. Denote $\mathrm{II}(w_1,w_2) = \langle Aw_1, w_2\rangle$, with $w_i \in T_{X(u,v)}M$. Then $A = \mathrm{II}\cdot\mathrm{I}^{-1}$. Fix the basis B of the tangent plane given by $X_u = \partial X(u,v)/\partial u$ and $X_v = \partial X(u,v)/\partial v$. Denote by $\{E,F,G\}$ and $\{e,f,g\}$ the coefficients of I and II with respect to B. Then

$$H = \epsilon\,\frac{1}{2}\,\frac{eG - 2fF + gE}{EG - F^2}, \qquad K = \epsilon\,\frac{eg - f^2}{EG - F^2}. \tag{7}$$

Here $N = X_u \times X_v / \sqrt{-\epsilon(EG - F^2)}$, where $\times$ is the cross-product $\times$ in $\mathbf{E}^3_1$ in $\mathbf{E}^3_1$. Recall that $W = EG - F^2$ is positive (resp. negative) if the immersion is spacelike (resp. timelike). Finally, recall that the cross-product satisfies $\langle u \times v, w \rangle = \det(u, v, w)$ for any vectors $u, v, w \in \mathbf{E}^3_1$. Then (7) can be rewritten as

$$H = -\frac{G \det(X_u, X_v, X_{uu}) - 2F \det(X_u, X_v, X_{uv}) + E \det(X_u, X_v, X_{vv})}{2(-\epsilon(EG - F^2))^{3/2}} = -\frac{H_1}{2(-\epsilon(EG - F^2))^{3/2}}, \tag{8}$$

$$K = -\frac{\det(X_u, X_v, X_{uu}) \det(X_u, X_v, X_{vv}) - \det(X_u, X_v, X_{uv})^2}{(EG - F^2)^2} = -\frac{K_1}{(EG - F^2)^2}. \tag{9}$$

From (8), we have

$$4H^2 \left| EG - F^2 \right|^3 - H_1^2 = 0. \tag{10}$$

4. Proof of Theorem 1.1

Consider a helicoidal non-degenerate surface generated by the graph of the polynomial $f(s) = \sum_{n=0}^{m} a_n s^n$, with $a_m \neq 0$. We distinguish three cases according to the causal character of the axis L.

4.1. The axis is timelike

Assume that $L = \langle (0, 0, 1) \rangle$. By Proposition 2.2, we can suppose that the generating curve γ is contained in the plane $y = 0$. If γ is not locally a graph on the x-axis, then γ is the vertical line $\gamma(s) = (r, 0, s)$, whose corresponding helicoidal surface is the Lorentzian cylinder of equation $x^2 + y^2 = r^2$. This surface has constant mean curvature $H = 1/(2r)$. Next assume that γ is given by $\gamma(s) = (s, 0, f(s))$, then

$$H = -\frac{s^2 f'(1 - f'^2) + s(s^2 - h^2) f'' - 2h^2 f'}{2(s^2(1 - f'^2) - h^2)^{3/2}}.$$

Consider separately the cases $H = 0$ and $H \neq 0$.

If $H = 0$, the numerator H_1 vanishes for any s, being $H_1 = 0$ a polynomial equation on the variable s. Then all coefficients must vanish. If $m \geq 2$, the leading coefficient corresponds to $s^2 f'^3$, that is, to s^{3m-1}. This coefficient is $-m^3 a_m^3$ and it implies $a_m = 0$, a contradiction. Thus $m < 2$. If $m = 0$, then $f(s) = a_0$ and $H = 0$. If $m = 1$, the leading coefficient of the numerator is $a_1\left(1 - a_1^2\right) = 0$. Thus $a_1 = \pm 1$. Now $H_1 = \pm 2h^2$, a contradiction again because we assumed that the pitch h is not zero.

Assume $H \neq 0$, then (10) is a polynomial equation on s and thus all the coefficients vanish. If $m \geq 2$, the degree of the polynomial is s^{6m}. The leading coefficient is $4H^2 m^6 a_m^6$, a contradiction. If $m = 1$, the expression (10) is a polynomial equation of degree 6, whose leading coefficient is $4H^2\left(1 - a_1^2\right)^3$. Thus $a_1 = \pm 1$. Using this value of a_1, $W = -h^2$. A new computation of (10) gives $4h^4\left(-1 + h^2 H^2\right) = 0$, thus $|H| = 1/h$.

Hence f is a constant function and either $H = 0$ or $f(s) = \pm s + a_0$ and $H \neq 0$. In the first case, the surface $X(s, t) = (s \cos t, s \sin t, ht + a_0)$ is the helicoid of first kind followed by a translation in the direction of the axis L.

4.2. The axis is spacelike

Assume that the axis is $L = \langle (1, 0, 0) \rangle$ and that the generating curve γ is contained in the plane $x = 0$ (Proposition 2.2). As in the timelike case, if γ is not locally a graph on the y-axis, then $\gamma(s) = (0, b, s)$. The helicoidal surface obtained from γ has constant mean curvature if $b = 0$, with $H = 0$, and the surface is the helicoid of third kind. Assume now $\gamma(s) = (0, s, f(s))$, then

$$H = -\frac{h(sf' - f)(f'^2 - 1) - h(h^2 - s^2 + f^2) f''}{2\left(-\epsilon(h^2 - s^2 + 2sff' - (h^2 + f^2) f'^2)\right)^{3/2}}.$$

First, suppose that $H = 0$. Then the numerator H_1 of H vanishes for any s. As $H_1 = 0$ is a polynomial equation on s, all the coefficients vanish. In this case, the leading coefficient corresponds to s^{3m-2}. This coefficient is $ha_m^3 m(m-1)^2$. Thus, if $m \geq 2$, we obtain $a_m = 0$, which is a contradiction. This implies $m \leq 1$. If $m = 0$, then $H_1 = ha_0$. This means that $a_0 = 0$. Suppose now that $m = 1$ then $H_1 = 0$ is $ha_0(1-a_1)^2 = 0$. We conclude that $a_0 = 0$ or $a_1 = \pm 1$. If $a_1 = \pm 1$, then $W = -a_0^2$, and thus, $a_0 \neq 0$.

Suppose now that H is a constant with $H \neq 0$. The polynomial equation on s given by (10) has as the leading coefficient $4H^2 m^6 a_m^{12}$ if $m \geq 2$, which corresponds to s^{12m-6}. Then $a_m = 0$, a contradiction. If $m = 1$, (10) is a polynomial equation of degree 6 and the corresponding leading coefficient is $4H^2(a_1^2 - 1)^6$. Then $a_1 = \pm 1$. But we know that in this case $H = 0$, a contradiction.

Therefore, we conclude that $H = 0$ and the degree of f is either 0 or 1. More precisely, the only possibilities are $f(s) = a_1 s$ or $f(s) = \pm s + a_0$, with $a_0, a_1 \in \mathbb{R}$, $a_0 \neq 0$. In the first case, we distinguish three possibilities.

1. If $|a_1| < 1$, the surface is a rigid motion of the helicoid of second kind. Let θ be a number such that $a_1 = \sin\theta/\cos\theta$ and define $\alpha(s) = (0, s/\cosh\theta, 0)$. We know that $G_{L,h}(\alpha)$ is the helicoid of second kind. On the other hand, for any $s, t \in \mathbb{R}$ we have $\phi_t(\gamma(s)) = \phi_{t+\theta}(\alpha(s)) - (h\theta, 0, 0)$. Then the surface S is
$$S = \{\phi_t(\gamma(s)) : s, t \in \mathbb{R}\} = G_{L,h}(\alpha) - (h\theta, 0, 0),$$
that is, S is a translation of the helicoid of second kind in the direction of the axis L.
2. If $|a_1| = 1$, then $W = 0$, and the surface is degenerated, which is not possible.
3. The case $|a_1| > 1$ is analogous to $|a_1| < 1$. The surface is the helicoid of third kind followed by a translation in the direction of the axis.

4.3. The axis is lightlike

By Proposition 2.2, we may assume that γ lies in the plane $\langle(1,0,1),(0,1,0)\rangle$. If γ is not a graph on the y-axis, then $\gamma(s) = (s, b, s)$ and the corresponding surface is the parabolic null cylinder with $H = 0$. Assume now that $\gamma(s) = (f(s), s, f(s))$, then
$$H = -\frac{4h^2(f' - 2sf'')}{2(4h\epsilon(s + hf'^2))^{3/2}}.$$
First assume that $H = 0$. Then the numerator is a polynomial on s that must be zero. The degree of this polynomial equation is $m-1$ if $m \geq 1$. The leading coefficient is $-4h^2 m a_m(2m-3)$. Then $a_m = 0$, which is a contradiction. If $m = 0$, then $f(s)$ is constant, $f(s) = a_0$, and $H = 0$.

Now assume that H is a non zero constant. The polynomial equation (10) is of degree $6m - 6$ if $m \geq 2$. The leading coefficient is $-256H^2h^6m^6a_m^6$. Thus $a_m = 0$, a contradiction. If $m = 1$, then (10) is a polynomial equation of degree 3, whose leading coefficient is $256h^3H^2$, a contradiction again. Finally, the case $m = 0$ leads to $H = 0$.

Therefore, for the lightlike case, the only possibility is $H = 0$ and f is a constant function, $f(s) = a_0$. The surface is parametrized as
$$X(s,t) = \left(st + h\left(\frac{t^3}{3} - t\right), s + ht^2, st + h\left(\frac{t^3}{3} + t\right)\right) + (a_0, 0, a_0), \qquad s, t \in \mathbb{R},$$
where $a_0 \in \mathbb{R}$, that is, it is a translation of the Cayley surface. This finishes the proof of Theorem 1.1. □

We end this section with a result on non-umbilical timelike surfaces with $H^2 - K = 0$. Recall that for a non-degenerate surface, there holds the identity $H^2 - \epsilon K \geq 0$. In the case when the surface is spacelike ($\epsilon = -1$), the Weingarten map A_p is diagonalizable and the equality $H(p)^2 + K(p) = 0$ means that p is umbilic, that is, A_p is proportional to the identity. However, if the surface is timelike ($\epsilon = 1$), A_p is not necessarily diagonalizable. In fact, it could be that $H(p)^2 - K(p) = 0$ and p is not umbilic. From Theorem 1.1, the helicoidal surface with timelike axis generated by $\gamma(s) = (s, 0, \pm s + a_0)$ is a surface with $|H| = 1/h$ and $K = 1/h^2$. Then $H^2 - K = 0$. On the other hand, the helicoidal surface with spacelike axis generated by $\gamma(s) = (0, s, \pm s + a_0)$ satisfies $H = K = 0$.

Theorem 4.1.

Consider a helicoidal timelike surface in $\mathbf{E}_1^3$ whose generating curve is the graph of a polynomial $f(s)=\sum_{n=0}^m a_n s^n$. If $H^2-K=0$, then $m\leq 1$. Moreover, up to a rigid motion of $\mathbf{E}_1^3$, the parametrization of the surface is one the following.

1. *If the axis is timelike,*
$$X(s,t)=(s\cos t,\, s\sin t,\, \pm s+a_0+ht),\qquad s,t\in\mathbb{R},$$
where $a_0\in\mathbb{R}$, $|H|=1/h$ and $K=1/h^2$.
2. *If the axis is spacelike,*
$$X(s,t)=\big(ht,\,(\pm s+a_0)\sinh t+s\cosh t,\,(\pm s+a_0)\cosh t+s\sinh t\big),\qquad s,t\in\mathbb{R},$$
where $a_0\neq 0$ and $H=K=0$.
3. *If the axis is lightlike, then the surface is the parabolic null cylinder*
$$X(s,t)=\left(s+bt+h\left(\frac{t^3}{3}-t\right),\, b+ht^2,\, s+bt+h\left(\frac{t^3}{3}+t\right)\right),\qquad s,t\in\mathbb{R},$$
where $b\in\mathbb{R}$ and $H=K=0$.

Proof. From (8) and (9), and the fact that $W<0$, the identity $H^2-K=0$ is equivalent to $H_1^2-4WK_1=0$. As the generating curve is the graph of a polynomial on s, this equation can be rewritten as $P(s)=\sum_{n=0}^k A_n s^n=0$. Hence all coefficients A_n must be zero. We distinguish the three cases depending on the causal character of the axis.

1. *The axis is timelike.* The Lorentzian cylinder does not satisfy $H^2-K=0$. Thus we assume that $\gamma(s)=(s,0,f(s))$. The equation $H_1^2-4WK_1=0$ is written as

$$\Big(\big(-2h^2+s^2\big)f-s^2f'^3+s\big(s^2-h^2\big)f''\Big)^2-4\big(h^2-s^2+s^2f'^2\big)\big(h^2-s^3f'f''\big)\;=0.$$

If $m\geq 2$, the leading coefficient of P comes from $s^4f'^6$ which is $m^6a_m^6$, a contradiction. If $m=1$, the degree of P is $k=4$, and the leading coefficient is $A_4=a_1^2\big(1-a_1^2\big)^2$. Hence we obtain $a_1=\pm1$. In this case, $|H|=1/h$, $K=1/h^2$, $W=-h^2$ and the Weingarten map of the surface is

$$\begin{pmatrix}1/h & 0\\ -1 & 1/h\end{pmatrix}.$$

If $m=0$, the surface is the helicoid of first kind, which does not satisfy the relation $H^2-K=0$.

2. *The axis is spacelike.* If γ is not a graph on the y-axis, then $\gamma(s)=(0,b,s)$, but the surface generated by γ does not satisfy $H^2-K=0$. Suppose that $\gamma(s)=(0,s,f(s))$, then $H_1^2-4WK_1=0$ is

$$\begin{aligned}&h\big(sf'-sf'^3+f\big(f'^2-1\big)+\big(h^2-s^2\big)f''+f^2f''\big)^2\\&\qquad+4\big((f-sf')^2+\big(h^2-s^2+f^2\big)\big(-1+f'^2\big)\big)\big(hf''(f-sf')-h(f'^2-1)^2\big)\;=0.\end{aligned}$$

If $m\geq 2$, the polynomial equation is of degree $k=8m-6$ and the leading coefficient is $-4hm^6a_m^6$. This gives a contradiction. Assume $m=1$. The degree is $k=2$ and the leading coefficient is $A_2=4h^2\big(1-a_1^2\big)^4$. Then $a_1=\pm1$. The surface satisfies $H=K=0$ and the Weingarten map is

$$\begin{pmatrix}0 & 0\\ -h/a_0 & 0\end{pmatrix}.$$

In case $m=0$, the equation $P=0$ reduces to $-4s^2+4h^2+a_0^2=0$, obtaining a contradiction again.

3. *The axis is lightlike.* We point out that if $\gamma(s)=(s,b,s)$ the surface is the parabolic null cylinder with $H=K=0$. Assume that $\gamma(s)=(f(s),s,f(s))$, then $H_1^2-4WK_1=0$ reduces to

$$h(f'-2sf'')^2-4\big(s+hf'^2\big)(1+2hf'f'')=0.$$

When $m\geq 2$, the degree of P is $k=4m-5$ and it comes from $-8h^2f'^3f''$. The leading coefficient is $-8h^2m^4(m-1)a_m^4$, a contradiction. If $m=1$, the equation reduces to $3ha_1^2+4s=0$, which leads to a contradiction again. If $m=0$, the equation is $hs=0$, a contradiction. □

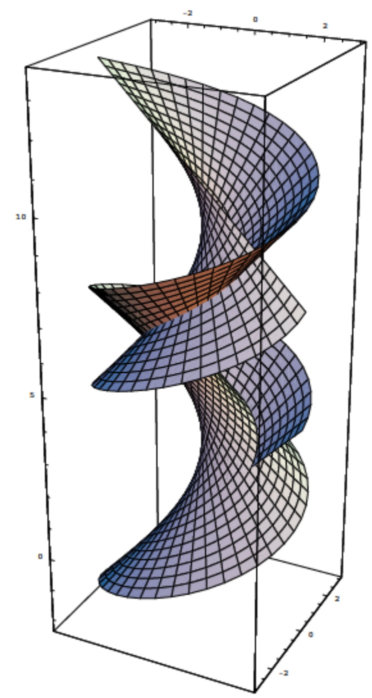
Remark 4.2.

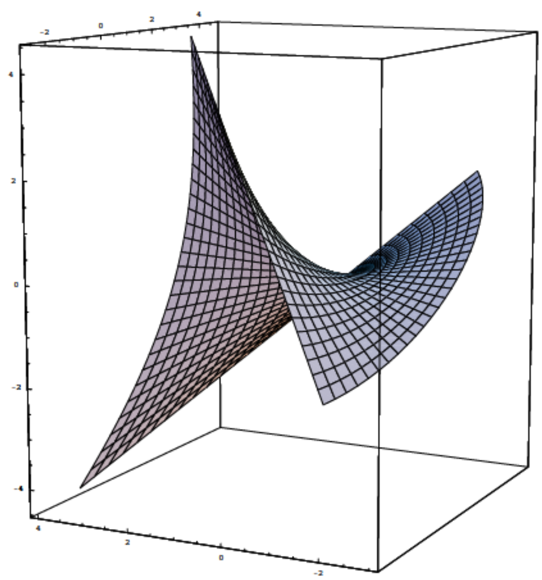
The helicoidal surfaces that appear in Theorem 4.1 are generated by lightlike straight-lines. Both surfaces are ruled and [2, Theorem 2] asserts that if a ruling is lightlike, then $H^2=K$, such as it occurs in our situation.

Remark 4.3.

The minimal timelike surface $X(s,t)=(s\cos t, s\sin t, \pm s+a_0+ht)$ is different from the three helicoids and the Cayley surface. For the choice of $a_0=0$, this surface appears in [5, Example 5.3]. On the other hand, the two surfaces that appear in Theorem 4.1 are *linear* Weingarten surfaces, that is, they satisfy a relation of type $aH+bK=c$, with $a,b,c\in\mathbb{R}$.

Figure 1. The timelike $X(s,t)=(s\cos t, s\sin t, \pm s+a_0+ht)$ for $a_0=h=1$ (left). The surface $X(s,t)=\big(st+h\big(t^3/3-t\big)+a_0, s+ht^2, st+h\big(t^3/3+t\big)+a_0\big)$ for $a_0=h=1$ (right).





5. Proof of Theorem 1.2

Consider a helicoidal surface generated by a Lorentzian circle. We distinguish the three cases depending on the causal character of the axis.

5.1. The axis is timelike

Consider the axis $L = \langle(0,0,1)\rangle$ and the generating curve $\gamma(s) = (x(s), 0, z(s))$. Here the circle γ lies in the timelike plane Π of equation $y = 0$. Then the parametrization of γ is, up a rigid motion of Π, the curve $x^2 - z^2 = \pm r^2$. We take the first possibility, that is, the circle of equation $x^2 - z^2 = r^2$. The case $x^2 - z^2 = -r^2$ is analogous. Thus

$$\gamma(s) = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} r\cosh s \\ 0 \\ r\sinh s \end{pmatrix} + \begin{pmatrix} \lambda \\ 0 \\ \mu \end{pmatrix} = \big(\lambda + r\cosh(s+\theta), 0, \mu + r\sinh(s+\theta)\big),$$

with $\theta, \lambda, \mu \in \mathbb{R}$. Using the parametrization $X(s,t) = \phi_t(\gamma(s))$, we compute the mean curvature considering separately the cases $H = 0$ and $H \neq 0$.

1. If $H = 0$, then $H_1 = 0$, which is equivalent to

$$\sum_{n=0}^{3} A_n \cosh(n(s+\theta)) = 0.$$

As the functions $\{\cosh(n(s+\theta)) : 0 \leq n \leq 3\}$ are linearly independent, then $A_n = 0$ for all $0 \leq n \leq 3$. But the leading coefficient is $A_3 = r^3\big(h^2 + r^2\big)/2$, a contradiction.

2. Assume that H is a non-zero constant. Then the identity (10) writes as

$$\sum_{n=0}^{6} A_n \cosh(n(s+\theta)) = 0.$$

A straightforward computation gives

$$A_6 = -\frac{1}{8} r^6 \big(h^2 + r^2\big)^2 \big(\pm 1 + H^2\big(h^2 + r^2\big)\big),$$

where $\pm 1 + H^2\big(h^2 + r^2\big)$ depends on whether the surface is spacelike or timelike. If the choice is $1 + H^2\big(h^2 + r^2\big)$, we get a contradiction. In the case $-1 + H^2\big(h^2 + r^2\big)$, $H^2 = 1/\big(h^2 + r^2\big)$ and $A_5 = \lambda r^7\big(h^2 + r^2\big)/4$. Then $\lambda = 0$. But $A_2 = 3h^4 r^6/2$ and $A_2 = 0$ yield a contradiction.

As a conclusion, the axis cannot be timelike.

5.2. The axis is spacelike

Assume that $L = \langle(1,0,0)\rangle$ and the generating curve $\gamma(s) = (0, y(s), z(s))$ lies in the plane Π of equation $x = 0$. As in the previous case, the plane Π is timelike and thus, the Lorentzian circles are rigid motions of the circle $y^2 - z^2 = \pm r^2$. Without loss of generality, we may suppose $y^2 - z^2 = r^2$. Then γ writes as

$$\gamma(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 0 \\ r\cosh s \\ r\sinh s \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda \\ \mu \end{pmatrix} = \big(0, \lambda + r\cosh(s+\theta), \mu + r\sinh(s+\theta)\big),$$

with $\theta, \lambda, \mu \in \mathbb{R}$. The parametrization of the surface is given by $X(s,t) = \phi_t(\gamma(s))$.

1. Case $H = 0$. Then $H_1 = 0$ writes as

$$hr^2\big(-\lambda^2 + \mu^2 + h^2 - r\lambda\cosh(s+\theta) + r\mu\sinh(s+\theta)\big) = 0.$$

As the functions $\cosh(s+\theta)$ and $\sinh(s+\theta)$ are linearly independent, we deduce that their coefficients vanish, that is, $\lambda=\mu=0$. Now $H=0$ is equivalent to $h^3r^2=0$, contradiction.

2. Case $H\neq 0$. Equation (10) has the form

$$\sum_{n=0}^{6}\big(A_n\cosh(n(s+\theta))+B_n\sinh(n(s+\theta))\big)=0.$$

Thus $A_n=B_n=0$ for all $0\leq n\leq 6$. Here we have

$$A_6=-\frac{1}{8}\left(\lambda^2+\mu^2\right)\left(\lambda^4+14\lambda^2\mu^2+\mu^4\right)H^2r^6=0,$$

hence $\lambda=\mu=0$. So (10) reduces to $h^6r^4\left(-1+4H^2r^2\right)=0$, that is, $|H|=1/(2r)$. Therefore, the generating curve is $\gamma(s)=(0,r\cosh(s+\theta),r\sinh(s+\theta))$, that is, the circle of equation $y^2-z^2=r^2$ in the plane Π.

We conclude that if the axis is spacelike, then the generating curve is a Lorentzian circle centered at the axis.

5.3. The axis is lightlike

Consider the axis $L=\langle(1,0,1)\rangle$. A helicoidal surface with axis L has the generating curve in the plane of equation $x-z=0$. A Lorentzian circle γ in this plane is a rigid motion of the circle $s\mapsto cs(0,1,0)+c/2s^2(1,0,1)$, $c\neq 0$. Then

$$\gamma(s)=\begin{pmatrix}1-\theta^2/2 & \theta & \theta^2/2\\ -\theta & 1 & \theta\\ -\theta^2/2 & t & 1+\theta^2/2\end{pmatrix}\begin{pmatrix}cs^2/2\\ cs\\ cs^2/2\end{pmatrix}+\begin{pmatrix}\lambda\\ \mu\\ \lambda\end{pmatrix},$$

with $\theta,\lambda,\mu\in\mathbb{R}$.

1. Suppose $H=0$. Then $H_1=0$ is $4c^2h^2(-2\mu+c\theta-chs)=0$. Thus $c^2h^2=0$, a contradiction.

2. If $H\neq 0$, then (10) is a polynomial equation on s of degree 6. The leading coefficient is $-256c^6h^6H^2$. This gives a contradiction again.

This means that the case when L is lightlike cannot occur. This finishes the proof of Theorem 1.2.

6. Proof of Theorem 1.3

We give the proof only for the case when the generating curve γ is the graph of a polynomial; if γ is a circle the arguments are similar to the ones given in Section 5. From (9), we have $KW^2+K_1=0$. Since the generating curve is the graph of a polynomial $f(s)=\sum_{n=0}^m a_ns^n$, the above expression writes as

$$P(s)=\sum_{n=0}^{k}A_ns^n=0. \tag{11}$$

Therefore, all coefficients A_n must be zero. We distinguish the three cases of axis.

1. *The axis is timelike.* If the curve γ is not a graph on the x-axis, the surface is the Lorentzian cylinder of equation $x^2+y^2=r^2$ whose Gauss curvature is $K=0$. Suppose the general case that the surface is parametrized as (1). Then

$$K=\frac{h^2-s^3f'f''}{(-h^2+s^2-s^2f'^2)^2}.$$

If $K = 0$ and if $m \geq 2$, the degree of P is $2m$, whose leading coefficient is $-m^2(m-1)a_m^2$ which is not possible. If $m \leq 1$, (11) reduces to $h^2 = 0$, a contradiction.

When K is a non-zero constant, the degree of P is $k = 4m$ if $m \geq 2$. The leading coefficient is $m^4 a_m^4 K$, a contradiction. If $m = 1$, the degree of P is $k = 4$, with $A_4 = K\left(1 - a_1^2\right)^2$. Hence we obtain $a_1 = \pm 1$. With this value of a_1, we get $P = h^2\left(-1 + h^2 K\right)$ and so $K = 1/h^2$. When $m = 0$, the degree of P is 4 again, with $A_4 = K$, a contradiction.

2. *The axis is spacelike.* As the helicoid of third kind has not constant Gauss curvature, we use (2) and the expression of K is

$$K = \frac{h\left(h(1-f'^2)^2 - h(f - sf')f''\right)}{\left((f - sf')^2 + (h^2 - s^2 + f^2)(1 - f'^2)\right)^2}.$$

If $K = 0$ and if $m \geq 2$, the degree of P is $k = 4m - 4$ with the leading coefficient $-h^2 m^4 a_m^4$, a contradiction. If $m = 1$, then $P = h^2\left(1 - a_1^2\right)$. Thus $a_1 = \pm 1$, and $a_0 \neq 0$ in order to ensure $W \neq 0$; if $m = 0$, $P = -h^2 = 0$, a contradiction.

Assume that $K \neq 0$. If $m \geq 2$, the degree of P is $8m - 4$ with the leading coefficient $m^4 a_m^8 K$, a contradiction. When $m = 1$, $k = 4$ with $A_4 = K\left(1 - a_1^2\right)^4$. Then $a_1 = \pm 1$. Now P reduces to $K a_0^4$. Then $a_0 = 0$, but now $W = 0$, contradiction. When $m = 0$, the degree of P is $k = 4$ with $A_4 = K$, a contradiction.

3. *The axis is lightlike.* If γ is not a graph on the y-axis, the corresponding surface is the parabolic null cylinder, which it is a surface with $K = 0$. In general, the value of K using (3) is

$$K = \frac{1 + 2hf'f''}{4(s + hf'^2)^2}.$$

Let $K = 0$. If $m \geq 2$, the degree of P is $2m - 3$ whose leading coefficient is $-8h^3 m^2(m-1)a_m^2$, a contradiction. If $m \leq 1$, $P = -4h^2$, a contradiction again.

If K is a non-zero constant, the degree of P is $k = 4m - 4$ if $m \geq 2$. The leading coefficient is $16h^4 m^4 a_m^4 K$: contradiction. If $m \leq 1$, the degree of P is $k = 2$, with $A_2 = 16h^2 K$, a contradiction again.

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