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A numerical solution of the MEW equation using sextic B-splines

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Abstract. In this article, a numerical solution of the modified equal width wave (MEW) equation, based on subdomain method using sextic B-spline is used to simulate the motion of single solitary wave and interaction of two solitary waves. The three invariants of the motion are calculated to determine the conservation properties of the system. L_2 and L_∞ error norms are used to measure differences between the analytical and numerical solutions. The obtained results are compared with some published numerical solutions. A linear stability analysis of the scheme is also investigated.

Keywords: Subdomain; Finite element method; MEW equation; Sextic B-spline; Solitary waves.

Mathematics Subject Classification 2010: 97N40, 65N30, 65D07, 76B25, 74S05.

1 Introduction

This study is concerned with applying the sextic B-spline function to develop a numerical method for approximation of the MEW equation of the form

$$U_t + 3U^2U_x - \mu U_{xxt} = 0, \tag{1.1}$$

with the boundary conditions

$$\begin{array}{lll} U(a,t) = 0, & U(b,t) = 0, & \\ U_x(a,t) = 0, & U_x(b,t) = 0, & t > 0, \end{array} \tag{1.2}$$

and the initial condition

$$U(x,0) = f(x) \quad a \le x \le b$$

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where t is time, x is the space coordinate, μ is a positive parameter, $U(x,t)$ is wave amplitude and $f(x)$ is a prescribed function. The equal width wave (EW) equation, which is an alternative form of the nonlinear dispersive waves to the well known Korteweg- de Vries (KdV) and regularized long wave (RLW) equation is a model non-linear partial differential equation used for the simulation of one-dimensional non-linear waves propagating in dispersive media. The EW equation represents a number of important physical phenomena among which are shallow water waves and ion acoustic plasma waves [2, 14]. MEW equation, which we discuss here, is related with the modified regularized long wave (MRLW)equation [1] and modified Korteweg-de Vries (MKdV) equation [7] is based upon the equal width wave (EW) equation. The modified equations are non-linear wave equations with cubic nonlinearities and have solitary wave solutions which are pulse like [14]. MEW equation has solitary wave solutions with both positive and negative amplitudes, all of which have the same width. The MEW equation with a limited set of boundary and initial conditions has an analytical solution like the EW equation. So, many numerical methods have been used for solving the MEW equation. Wazwaz [17] investigated the MEW equation and two of its variants by the tanh and the sine-cosine methods. Zaki [18,19] considered the solitary wave interactions for the MEW equation by Petrov-Galerkin method using quintic B-spline finite elements and obtained the numerical solution of the EW equation by using the least-squares method. Variational iteration method is introduced to solve the MEW equation by Junfeng Lu [12]. Esen [3, 4]applied a lumped Galerkin method based on quadratic B-spline finite elements have been used for solving the EW and MEW equation. A. Esen and S. Kutluay [5] studied a linearized implicit finite difference method in solving the MEW equation. Saka [16] proposed algorithms for the numerical solution of the MEW equation using quintic B-spline collocation method. T. Geyikli and S. Battal Gazi Karakoç [9, 10] solved the MEW equation by a collocation method using septic B-spline finite elements and using a Petrov-Galerkin finite element method with weight functions quadratic and element shape functions are cubic B-splines. D. J. Evans and K. R. Raslan [6] studied the generalized EW equation by using collocation method based on quadratic B-splines to obtain the numerical solutions of a single solitary waves, and the birth of solitons. Hamdi et al. [11] derived exact solitary wave solutions of the GEW equation.

In this work, Subdomain method is developed for the MEW equation using sextic B-spline function. The proposed method is shown to represent accurately the motion of single solitary wave and interaction of the two solitary waves. A linear stability analysis based on a Fourier method shows that the numerical scheme is unconditionally stable.

2 Sextic B-spline subdomain finite element method

The region $[a, b]$ is partitioned into uniformly sized finite elements of equal length h by the knots x_m such that $a = x_0 < x_1 \cdots < x_N = b$. Let $\phi_m(x)$ be sextic B-splines with knots at the points $x_m, m = 0, 1, ..., N$. The set of splines $\{\phi_{-3}, \phi_{-2}, \ldots, \phi_{N+1}, \phi_{N+2}\}$ forms a basis for functions defined over $[a, b]$. So a global approximation $U_N(x,t)$ to the exact solution $U(x,t)$ can be expressed in terms of the sextic B-splines as:

$$U_N(x,t) = \sum_{m=-3}^{N+2} \delta_m(t)\phi_m(x) \tag{2.1}$$

where δ_m are time dependent quantities to be determined from both boundary and weighted residual conditions. Each sextic B-spline covers seven elements so that each element $[x_m, x_{m+1}]$ is covered by seven splines. Sextic B-splines $\phi_m(x)$, $(m = -3(1)N+2)$, at the knots x_m which form a basis over the interval $[a,b]$ are defined by the relationships [15];

$$\phi_m(x) = \frac{1}{h^6}\begin{cases} (x-x_{m-3})^6, & x\in[x_{m-3},x_{m-2}],\\ (x-x_{m-3})^6 - 7(x-x_{m-2})^6, & x\in[x_{m-2},x_{m-1}],\\ (x-x_{m-3})^6 - 7(x-x_{m-2})^6 + 21(x-x_{m-1})^6, & x\in[x_{m-1},x_m],\\ (x-x_{m-3})^6 - 7(x-x_{m-2})^6 + 21(x-x_{m-1})^6 - 35(x-x_m)^6, & x\in[x_m,x_{m+1}],\\ (x-x_{m+4})^6 - 7(x-x_{m+3})^6 + 21(x-x_{m+2})^6, & x\in[x_{m+1},x_{m+2}],\\ (x-x_{m+4})^6 - 7(x-x_{m+3})^6, & x\in[x_{m+2},x_{m+3}],\\ (x-x_{m+4})^6, & x\in[x_{m+3},x_{m+4}],\\ 0, & otherwise, \end{cases} \tag{2.2}$$

where $h = (x_{m+1} - x_m)$. Using (2.1) and (2.2), the nodal values U and its $1^{st}, 2^{nd}$ and 3^{rd} derivatives at the knots x_m are obtained as follows:

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-3} + 57\delta_{m-2} + 302\delta_{m-1} + 302\delta_m + 57\delta_{m+1} + \delta_{m+2},\\ U'_m &= U'(x_m) = \tfrac{6}{h}(-\delta_{m-3} - 25\delta_{m-2} - 40\delta_{m-1} + 40\delta m + 25\delta_{m+1} + \delta_{m+2}),\\ U''_m &= U''(x_m) = \tfrac{30}{h^2}(\delta_{m-3} + 9\delta_{m-2} - 10\delta_{m-1} - 10\delta_m + 9\delta_{m+1} + \delta_{m+2}),\\ U'''_m &= U'''(x_m) = \tfrac{120}{h^3}(-\delta_{m-3} - \delta_{m-2} + 8\delta_{m-1} - 8\delta_m + \delta_{m+1} + \delta_{m+2}). \end{aligned} \tag{2.3}$$

A typical finite interval $[x_m, x_{m+1}]$ is mapped to the interval $[0,1]$ by local coordinates ξ related to the global coordinates

$$h\xi = x - x_m, \quad 0 \le \xi \le 1 \tag{2.4}$$

so the sextic B-spline shape functions over the element $[0,1]$ can be defined as $\phi^e = (\phi_{m-3}, \phi_{m-2}, \phi_{m-1}, \phi_m, \phi_{m+1}, \phi_{m+2}, \phi_{m+3})$,

$$\phi^e = \begin{cases} \phi_{m-3} = 1 - 6\xi + 15\xi^2 - 20\xi^3 + 15\xi^4 - 6\xi^5 + \xi^6,\\ \phi_{m-2} = 57 - 150\xi + 135\xi^2 - 20\xi^3 - 45\xi^4 + 30\xi^5 - 6\xi^6,\\ \phi_{m-1} = 302 - 240\xi - 150\xi^2 + 160\xi^3 + 30\xi^4 - 60\xi^5 + 15\xi^6,\\ \phi_m = 302 + 240\xi - 150\xi^2 - 160\xi^3 + 30\xi^4 + 60\xi^5 - 20\xi^6,\\ \phi_{m+1} = 57 + 150\xi + 135\xi^2 + 20\xi^3 - 45\xi^4 - 30\xi^5 + 156\xi^6,\\ \phi_{m+2} = 1 + 6\xi + 15\xi^2 + 20\xi^3 + 15\xi^4 + 6\xi^5 - 6\xi^6,\\ \phi_{m+3} = \xi^6. \end{cases} \tag{2.5}$$

Since all splines apart from $\phi_{m-3}(x), \phi_{m-2}(x), \phi_{m-1}(x), \phi_m(x), \phi_{m+1}(x), \phi_{m+2}(x), \phi_{m+3}(x)$ are zero over the element $[0,1]$. Approximation(2.2) over this element can be written in terms of basis functions (2.5) as

$$U_N(\xi,t) = \sum_{j=m-3}^{m+3} \delta_j(t)\phi_j(\xi)$$

where $\delta_{m-3}, \delta_{m-2}, \delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2}, \delta_{m+3}$ act as element parameters and B-splines $\phi_{m-3}(x), \phi_{m-2}(x), \phi_{m-1}(x), \phi_m(x), \phi_{m+1}(x), \phi_{m+2}(x)$ and $\phi_{m+3}(x)$ as element shape functions. Application of Subdomain method to the Eq.(1.1) with weight function

$$W_m(x) = \begin{cases} 1, & x \in [x_m, x_{m+1}], \\ 0, & otherwise \end{cases} \tag{2.6}$$

produces the weak form

$$\int_{x_m}^{x_{m+1}} 1.(U_t + 3U^2 U_x - \mu U_{xxt} dx) = 0. \tag{2.7}$$

Substituting the transformation (2.4) into weak form(2.7) and integrating Eq. (2.7) term by term with some manipulation by parts, leads to

$$\begin{array}{l} \frac{h}{7}(\dot{\delta}_{m-3} + 120\dot{\delta}_{m-2} + 1191\dot{\delta}_{m-1} + 2416\dot{\delta}_m + 1191\dot{\delta}_{m+1} + 120\dot{\delta}_{m+2} + \dot{\delta}_{m+3}) \\ +Z_m(-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1} + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}) \\ -\frac{4\mu}{h}(-\dot{\delta}_{m-3} - 25\dot{\delta}_{m-2} - 40\dot{\delta}_{m-1} + 40\dot{\delta}_m + 25\dot{\delta}_{m+1} + \dot{\delta}_{m+2}) = 0, \end{array} \tag{2.8}$$

where the dot denotes differentiation with respect to t and

$$Z_m = 3(\delta_{m-3} + 57\delta_{m-2} + 302\delta_{m-1} + 302\delta_m + 57\delta_{m+1} + \delta_{m+2})^2.$$

If time parameters δ_m and its time derivatives $\dot{\delta}_m$ in Eq. (2.8) are discretized by the Crank-Nicolson and forward difference approach respectively,

$$\delta = \frac{\delta_m^n + \delta_m^{n+1}}{2}, \qquad \dot{\delta}_m = \frac{\delta_m^{n+1} - \delta_m^n}{\Delta t}, \tag{2.9}$$

we obtain a recurrence relationship between two time levels n and $n+1$ relating two unknown parameters $\delta_i^{n+1}, \delta_i^n, \quad i = m-3, m-2, \ldots, m+3,$

$$\begin{array}{l} \alpha_{m1}\delta_{m-3}^{n+1} + \alpha_{m2}\delta_{m-2}^{n+1} + \alpha_{m3}\delta_{m-1}^{n+1} + \alpha_{m4}\delta_m^{n+1} + \alpha_{m5}\delta_{m+1}^{n+1} + \alpha_{m6}\delta_{m+2}^{n+1} + \alpha_{m7}\delta_{m+3}^{n+1} = \\ \alpha_{m7}\delta_{m-3}^n + \alpha_{m6}\delta_{m-2}^n + \alpha_{m5}\delta_{m-1}^n + \alpha_{m4}\delta_m^n + \alpha_{m3}\delta_{m+1}^n + \alpha_{m2}\delta_{m+2}^n + \alpha_{m1}\delta_{m+3}^n \end{array} \tag{2.10}$$

where

$$\begin{array}{ll} \alpha_{m1} = 1 - EZ_m - M, & \alpha_{m2} = 120 - 56EZ_m - 24M, \\ \alpha_{m3} = 1191 - 245EZ_m - 15M, & \alpha_{m4} = 2416 + 80M, \\ \alpha_{m5} = 1191 + 245EZ_m - 15M, & \alpha_{m6} = 120 + 56EZ_m - 24M, \\ \alpha_{m7} = 1 + EZ_m - M, & \quad m = 0, 1, \ldots, N-1, \end{array}$$

and

$$E = \frac{7\Delta t}{2h}, \qquad M = \frac{42\mu}{h^2}.$$

The system (2.10) consists of N linear equation in $N+6$ unknowns$(\delta_{-3}, \delta_{-2}, \ldots, \delta_{N+1}, \delta_{N+2})$. To get a solution of this system, we need six additional constraints. These are obtained from the boundary conditions (1.2). These conditions enable us elimination of the parameters $\delta_{-3}, \delta_{-2}, \delta_{-1}, \delta_N, \delta_{N+1}$ and δ_{N+2} from the system (2.10) which then becomes a matrix equation for the N unknowns $d = (\delta_0, \delta_1, \ldots, \delta_{N-1})$ of the form

$$Ad^{n+1} = Bd^n.$$

A lumped value for Z_m is obtained from $(U_m + U_{m+1})^2/4$ as

$$Z_m = \frac{3}{4}(\delta_{m-3}^n + 58\delta_{m-2}^n + 359\delta_{m-1} + 604\delta_m^n + 359\delta_{m+1}^n + 58\delta_{m+2}^n + \delta_{m+3}^n)^2.$$

The resulting system can be efficiently solved with a variant of the Thomas algorithm, and we need an inner iteration $\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ at each time step to cope with the non-linear term Z_m. A typical member of the matrix system (2.10) can be written in terms of the nodal parameters δ_m^n as

$$\begin{aligned} &\gamma_1\delta_{m-3}^{n+1} + \gamma_2\delta_{m-2}^{n+1} + \gamma_3\delta_{m-1}^{n+1} + \gamma_4 + \gamma_5\delta_{m+1}^{n+1} + \gamma_6\delta_{m+2}^{n+1} + \gamma_7\delta_{m+3}^{n+1} = \\ &\gamma_7\delta_{m-3}^n + \gamma_6\delta_{m-2}^n + \gamma_5\delta_{m-1}^n + \gamma_4 + \gamma_3\delta_{m+1}^n + \gamma_2\delta_{m+2}^n + \gamma_1\delta_{m+3}^n \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} &\gamma_1 = \alpha - \beta - \lambda, && \gamma_2 = 120\alpha - 56\beta - 24\lambda, \\ &\gamma_3 = 1191\alpha - 245\beta - 15\lambda, && \gamma_4 = 2416\alpha + 80\lambda, \\ &\gamma_5 = 1191\alpha + 245\beta - 15\lambda, && \gamma_6 = 120\alpha + 56\beta - 24\lambda, \\ &\gamma_7 = \alpha + \beta - \lambda \end{aligned}$$

and

$$\alpha = 1, \quad \beta = EZ_m, \quad \lambda = M, \qquad m = 0, 1, \ldots, N-1.$$

To start the recurrence relation system equation (2.10), initial parameters must be determined with the help of initial condition and six boundary conditions as follows:

$$\begin{aligned} U_N(x_m, 0) &= \delta_{m-3}^0 + 57\delta_{m-2}^0 + 302\delta_{m-1}^0 + 302\delta_m^0 + 57\delta_{m+1}^0 + \delta_{m+2}^0 = U(x_m, 0) \\ U_N'(a, 0) &= -\delta_{-3}^0 - 25\delta_{-2}^0 - 40\delta_{-1}^0 + 40\delta_0^0 + 25\delta_1^0 + \delta_2^0 = 0, \\ U_N''(a, 0) &= \delta_{-3}^0 + 9\delta_{-2}^0 - 10\delta_{-1}^0 - 10\delta_0^0 + 9\delta_1^0 + \delta_2^0 = 0, \\ U_N'''(a, 0) &= -\delta_{-3}^0 - \delta_{-2}^0 + 8\delta_{-1}^0 - 8\delta_0^0 + \delta_1^0 + \delta_2^0 = 0, \\ U_N'(b, 0) &= -\delta_{N-3}^0 - 25\delta_{N-2}^0 - 40\delta_{N-1}^0 + 40\delta_N^0 + 25\delta_{N+1}^0 + \delta_{N+2}^0 = 0, \\ U_N''(b, 0) &= \delta_{N-3}^0 + 9\delta_{N-2}^0 - 10\delta_{N-1}^0 - 10\delta_N^0 + 9\delta_{N+1}^0 + \delta_{N+2}^0 = 0, \\ U_N'''(b, 0) &= -\delta_{N-3}^0 - \delta_{N-2}^0 + 8\delta_{N-1}^0 - 8\delta_N^0 + \delta_{N+1}^0 + \delta_{N+2}^0 = 0. \end{aligned}$$

Eliminating $\delta_{-3}^0, \delta_{-2}^0, \delta_{-1}^0, \delta_N^0, \delta_{N+1}^0, \delta_{N+2}^0$ from the system (2.10) we get $N \times N$ matrix system of the form

$$W\delta^0 = K$$

where W is

$$W=\begin{bmatrix} 384 & 312 & 24 & & & & & & & & \\ \frac{2681}{9} & 358 & \frac{568}{9} & 1 & & & & & & & \\ \frac{512}{9} & 303 & \frac{2719}{9} & 57 & 1 & & & & & & \\ & 1 & 57 & 302 & 302 & 57 & 1 & & & & \\ & & & & & & & & & & \\ & & & & & & 1 & 57 & \frac{2719}{9} & 303 & \frac{512}{9} \\ & & & & & & & 1 & \frac{568}{9} & 358 & \frac{2681}{9} \\ & & & & & & & & 24 & 312 & 384 \end{bmatrix},$$

$\delta^0 = [\delta_0^0, \delta_1^0, \ldots, \delta_{N-1}^0]^T$ and $K = [U(x_0, 0), U(x_1, 0), \ldots, U(x_{N-1}, 0)]^T$. This matrix system can be solved efficiently by using a variant of Thomas algorithm.

2.1 Stability analysis

The stability analysis is based on the Von Neumann theory in which the growth factor of a typical Fourier mode

$$\delta_j^n = \xi^n e^{ijkh} \tag{2.12}$$

where k is mode number and h the element size, is determined for a linearization of the numerical scheme. To apply the stability analysis, the MEW equation needs to be linearized by assuming that the quantity U in the non-linear term U^2U_x is locally constant. Substituting the equation (2.12) into the scheme (2.11) we have

$$g = \frac{a - ib}{a + ib}, \tag{2.13}$$

where

$$\begin{aligned} a &= 1208 + 40\lambda + (1191 - 15\lambda)\cos(kh) + (120 - 24\lambda)\cos(2kh) + (1 - \lambda)\cos(3kh), \\ b &= 245\beta\sin(kh) + 56\beta\sin(2kh) + \beta\sin(3kh). \end{aligned} \tag{2.14}$$

Taking the modulus of equation(2.13) gives $|g| = 1$, therefore we find that the scheme (2.11) is unconditionally stable.

3 Numerical examples and results

We obtain the numerical solutions of the MEW equation for two problems: the motion of single solitary wave and interaction of two solitary waves. All numerical calculations were performed in double precision arithmetic on a Pentium 4 PC using a Fortran compiler. We use error norms L_2 and L_∞ defined by

$$L_2 = \left\|U^{exact} - U_N\right\|_2 \simeq \sqrt{h\sum_{J=1}^{N}\left|U_j^{exact} - (U_N)_j\right|^2},$$

and

$$L_\infty = \left\|U^{exact} - U_N\right\|_\infty \simeq \max_j \left|U_j^{exact} - (U_N)_j\right|,$$

to measure differences between analytical and numerical solutions and so to show how good the numerical scheme predicts the position and amplitude of the solution as the simulation proceeds. The MEW equation possesses only three conservation laws related with mass, momentum and energy given by [8] in the following manners respectively:

$$\begin{aligned}
&I_1 = \textstyle\int_a^b U dx \simeq h \sum_{J=1}^N U_j^n,\\
&I_2 = \textstyle\int_a^b [U^2 + \mu (U_x)^2] dx \simeq h \sum_{J=1}^N [(U_j^n)^2 + \mu\, (U_x)_j^n],\\
&I_3 = \textstyle\int_a^b U^4 dx \simeq h \sum_{J=1}^N (U_j^n)^4,
\end{aligned}$$

Olver [13] has shown that EW equation have just three such laws and this may also be true of the MEW equation also.

3.1 The motion of single solitary wave

For this problem, we consider solitary wave solution of Eq. (1.1)

$$U(x,t) = A \sec h(k[x - x_0 - vt]), \quad k = \sqrt{\frac{1}{\mu}}, \quad v = \frac{A^2}{2}.$$

This equation represents a single solitary wave of magnitude A, initially centered on x_0 and moving to the right with a constant velocity v. The initial condition

$$U(x,0) = A \sec h(k[x - x_0])$$

with constants $\mu = 1, x_0 = 30$ and boundary conditions $U \to 0$ as $x \to \pm\infty$ are used to coincide with earlier articles. For this problem the analytical values of the invariants are [18]

$$I_1 = \frac{A\pi}{k}, \quad I_2 = \frac{2A^2}{k} + \frac{2\mu k A^2}{3}, \quad I_3 = \frac{4A^4}{3k}. \tag{3.1}$$

For the numerical simulation of this problem we have used the parameters $h = 0.1,\ \Delta t = 0.05,\ \mu = 1,\ x_0 = 30,\ A = 0.25$ through the interval $0 \leq x \leq 80$. The analytical values of invariants are $I_1 = 0.7853982$, $I_2 = 0.1666667$, $I_3 = 0.0052083$. Program was run to time $t = 20$ to record the error norms L_2, L_∞ and conserved quantities I_1, I_2, I_3. Table 1 displays a comparison of the values of the invariants and error norms obtained by the present method with those obtained some earlier methods [4–6,10] at time $t = 20$. It is clear from the table that the error norms obtained by the present method are smaller than other methods [4–6, 10] and agreement between analytical and numerical solutions is excellent. At time $t = 20$, the difference between the analytical and numerical values of the conservation quantities are $\Delta I_1 = \Delta I_2 = 1 \times 10^{-7}$ and $\Delta I_3 = 0$. Invariants I_1, I_2 and I_3 change by less than $1 \times 10^{-6}\%$, $2 \times 10^{-6}\%$, $5 \times 10^{-6}\%$ throught the run, respectively. So the quantities I_1, I_2, I_3 remain constants during the computer run. Fig.1 shows that the proposed method perform the motion of propagation of a solitary wave satisfactorily, which moves to the right at a constant speed and preserves its amplitude and shape with increasing time as expected. Amplitude is 0.25 at $t = 0$ which is located at $x = 30$, while it is 0.249922 at $t = 20$ which is located at $x = 30.6$. The absolute difference in amplitudes at times $t = 0$ and $t = 20$ is 7.8×10^{-5} so that there is a little change between amplitudes.

We have also considered first problem for different values of the amplitude at time step of $t = 0.01$ and $h = 0.1$. In Table 2 the error norms and invariants are listed for $A = 0.25, 0.5, 0.75, 1$. This method is compared with Ref. [5] and the comparison of error norms show that the present method provide better results. Fig.2 shows the solutions of the single solitary wave with $h = 0.1, \Delta t = 0.01$ for different values of amplitude A at time $t = 20$. It is clear that the soliton moves to the right at a constant speed and almost preserve its amplitude and shape with an increasing of time, as expected.

Table 1: Invariants and error norms for single solitary wave with $h = 0.1$, $\Delta t = 0.05$, $A = 0.25$ and $0 \leq x \leq 80$.

t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	0.7853966	0.1666664	0.0052083	0.0000000	0.00000000
5	0.7853966	0.1666664	0.0052083	0.0129274	0.0077963
10	0.7853966	0.1666663	0.0052083	0.0258644	0.0157807
15	0.7853967	0.1666663	0.0052083	0.0388122	0.0239359
20	0.7853967	0.1666663	0.0052083	0.0517742	0.0321145
20 [4]	0.7853898	0.1667614	0.0052082	0.0796940	0.0465523
20 [5]	0.7853977	0.1664735	0.0052083	0.2692812	0.2569972
20 [10]	0.7853967	0.1666663	0.0052083	0.0801465	0.0461218
20 [6]	0.7849545	0.1664765	0.0051995	0.2905166	0.2498925

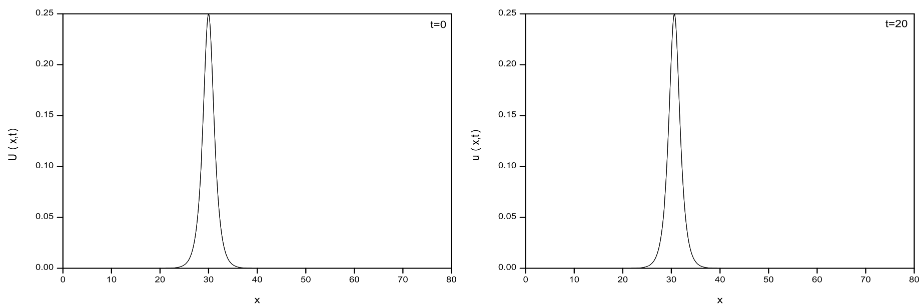

Figure 1: The motion of a single solitary wave with $h = 0.1, \Delta t = 0.05$ at $t = 0$ and $t = 20$.

The pointwise rates of convergence in space sizes h_m are calculated with the following formula respectively [4];

$$order = \frac{log_{10}(|U^{exact} - U^{num}_{h_m}|/|U^{exact} - U^{num}_{h_{m+1}}|)}{log_{10}(h_m/h_{m+1})}.$$

Table 2: Invariants and error norms for single solitary wave with different amplitudes, $h = 0.1$, $\Delta t = 0.01$ and $0 \leq x \leq 80$.

A	t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
	0	0.7853966	0.1666664	0.0052083	0.0000000	0.0000000
	5	0.7853966	0.1666664	0.0052083	0.0129274	0.0077963
0.25	10	0.7853966	0.1666663	0.0052083	0.0258644	0.0157807
	15	0.7853967	0.1666663	0.0052083	0.0388122	0.0239359
	20	0.7853967	0.1666663	0.0052083	0.0517742	0.0321145
	20 [5]	-	-	-	0.2692249	0.2569562
	0	1.5707932	0.6666654	0.0833330	0.0000000	0.0000000
	5	1.5707931	0.6666654	0.0833330	0.1036264	0.0642764
0.5	10	1.5707931	0.6666653	0.0833330	0.2079707	0.1320856
	15	1.5707930	0.6666653	0.0833330	0.3137330	0.2015173
	20	1.5707930	0.6666653	0.0833330	0.4211132	0.2711978
	20 [5]	-	-	-	1.82660590	1.4575680
	0	2.3561897	1.4999972	0.4218734	0.0000000	0.0000000
	5	2.3561896	1.4999970	0.4218733	0.3525506	0.2248542
0.75	10	2.3561895	1.4999970	0.4218733	0.7157735	0.4603748
	15	2.3561895	1.4999969	0.4218733	1.0944557	0.6980905
	20	2.3561895	1.4999969	0.4218733	1.4802085	0.9359489
	20 [5]	-	-	-	4.3957110	3.0917930
	0	3.1415863	2.6666616	1.3333283	0.0000000	0.0000000
	5	3.1415860	2.6666613	1.3333279	0.8522712	0.5483568
1.0	10	3.1415860	2.6666612	1.3333279	1.7589319	1.1165885
	15	3.1415860	2.6666612	1.3333279	2.6847470	1.6852944
	20	3.1415860	2.6666612	1.3333279	3.6129701	2.2540155
	20 [5]	-	-	-	8.2853140	5.6821310

Computations are carried out with different spatial step sizes to evaluate the point rates of convergence in space. In Table 3 the time step is kept fixed at $\Delta t = 0.05$ and $h = 0.8, 0.4, 0.2, 0.1, 0.05, 0.025$ to calculate spatial rate of convergence. It can observed from the Table 3 that the convergence rates when $\Delta t = 0.05$ is fixed are good for the space sizes.

Table 3: Space rate of convergence at $t = 20$, $\Delta t = 0.05$, $A = 0.25$, $0 \leq x \leq 80$.

h_m	$L_2 \times 10^3$	order	$L_\infty \times 10^3$	order
0.8	3.93871952	-	3.16133286	-
0.4	0.81179451	2.27854019	0.50991029	2.63221758
0.2	0.20571238	1.98048559	0.12802658	1.99379920
0.1	0.05183664	1.98858450	0.03215247	1.99344276
0.05	0.01324614	1.96840019	0.00810867	1.98739236
0.025	0.00325155	2.02637249	0.00204993	1.98389065

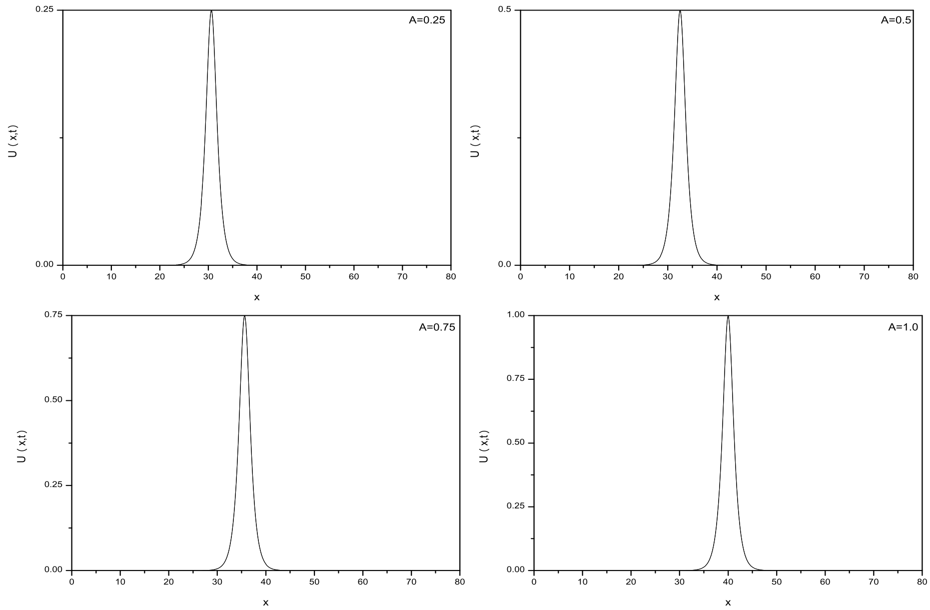

Figure 2: Single solitary wave solutions for various values of A at $t = 20$.

3.2 Interaction of two solitary waves

Secondly, interaction of two solitary waves is studied by using the initial condition

$$U(x, 0) = \sum_{j=1}^{2} A_j \sec h(k[x - x_j]), k = \sqrt{\frac{1}{\mu}}, \tag{3.2}$$

together with boundary conditions $U \to 0$ as $x \to \pm\infty$. The initial condition (3.2) represents two solitary waves, one with amplitude A_1 placed initially at $x = x_1$ and the second with amplitude A_2 placed at $x = x_2$. To ensure an interaction of two solitary waves we have used the parameters $h = 0.1,\ \mu = 1, \Delta t = 0.025,\ A_1 = 1,\ x_1 = 15,\ A_2 = 0.5,\ x_2 = 30$ over the interval $0 \leq x \leq 80$ to coincide with those used by Esen [4].The analytical invariants can be found as

$$\begin{aligned} I_1 &= \pi(A_1 + A_2) = 4.7123889, \qquad I_2 = \frac{8}{3}(A_1^2 + A_2^2) = 3.3333333, \\ I_3 &= \frac{4}{3}(A_1^4 + A_2^4) = 1.4166667. \end{aligned} \tag{3.3}$$

The experiment was run from $t = 0$ to $t = 80$ to allow the interaction in order to take place. Table 5 compares the computed values of the invariants of the two solitary waves with results from [4]. It is clear that the obtained values of the invariants are satisfactorily constant during the computer run. The absolute difference between the values of the invariants obtained by the present method at times $t = 0$ and $t = 55$

are $\Delta I_1 = 1.1 \times 10^{-6}, \Delta I_2 = 1.38 \times 10^{-5},\ \Delta I_3 = 2.4 \times 10^{-6}$ whereas they are $\Delta I_1 = 2.48 \times 10^{-4}, \Delta I_2 = 2.04 \times 10^{-4}, \Delta I_3 = 1.17 \times 10^{-4}$ in Ref. [4]. The changes in these quantities I_1, I_2, I_3 are less than $8 \times 10^{-6}\%,\ 3.9 \times 10^{-4}\%,\ 1.8 \times 10^{-4}\%$ during the run so that the numerical algorithm has good conservation properties. Fig.3 illustrates the behavior of the interaction of two positive solitary waves. It is observed from the Fig.3, at $t = 0$ the wave with larger amplitude is on the left of the second wave with smaller amplitude. Since the taller wave moves faster than the shorter one, it catches up and collides with the shorter one at $t = 35$ and then moves away from the shorter one as time increases. When the interaction is completed we get the Fig.3 at $t = 80$. At $t = 80$, the amplitude of larger waves is 1.000331 at the point $x = 56.9$ whereas the amplitude of the smaller one is 0.498729 at the point $x = 37.7$. It is found that the absolute difference in amplitude is 0.331×10^{-3} for the smaller wave and 0.127×10^{-2} for the larger wave for this algorithm. At $t = 80$, we saw an oscillation of small amplitude trailing behind the solitary waves. To see this oscillation the scale of figure at $t = 80$ magnified as shown in Fig.4. It is clearly seen from the Fig. 4 that an oscillation of the small amplitude is trailing behind the solitary waves.

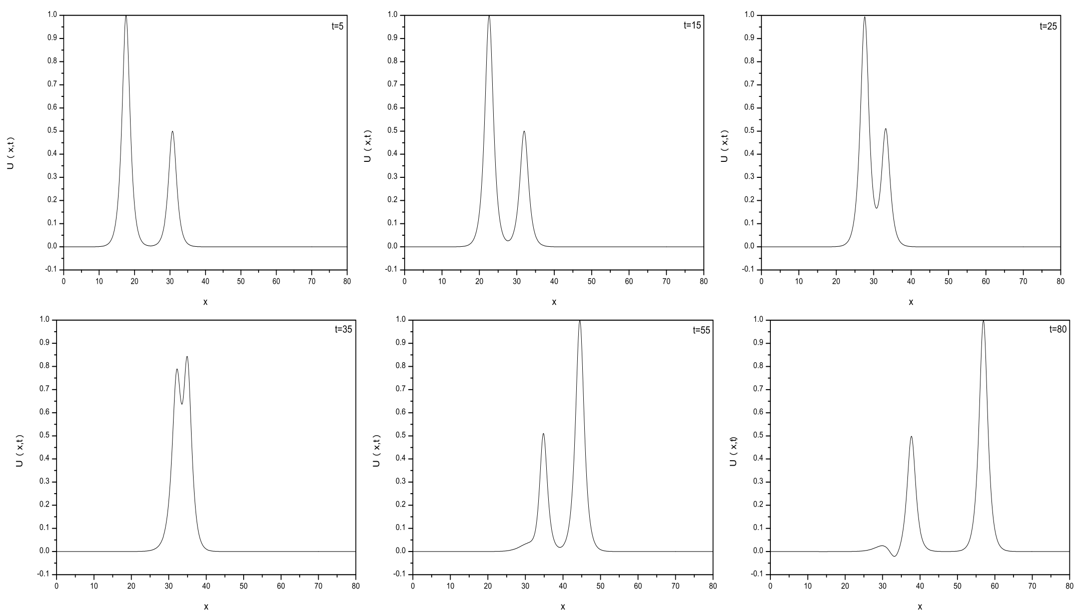

Figure 3: Interaction of two solitary waves with $A_1 = 1, A_2 = 0.5, 0 \leq x \leq 80$.

We have also studied the interaction of two solitary waves with the parameters $\mu = 1, x_1 = 15, x_2 = 30, A_1 = -2, A_2 = 1, h = 0.1$ and $\Delta t = 0.025$ in the range $0 \leq x \leq 150$. The experiment was run from $t = 0$ to $t = 55$ to allow the interaction to take place. Fig.5 shows the development of the solitary wave interaction. As is clearly seen from the Fig.5, at $t = 0$ a wave with negative amplitude is on the left of another wave with positive amplitude. The larger wave with the negative amplitude catches up with the smaller one with the positive amplitude as the time increases. Table 6 displays a

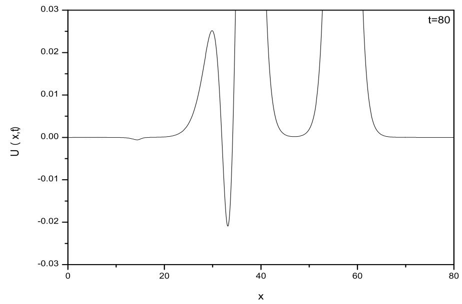

Figure 4: An expanded vertical scale of Fig.3 at t=80.

Table 4: Comparison of invariants for the interaction of two solitary waves with results from [4] with $h = 0.1, \Delta t = 0.025$ in the region $0 \leq x \leq 80$.

	Present method			Galerkin method [4]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	4.7123733	3.3333294	1.4166643	4.7123884	3.3352890	1.4166697
5	4.7123725	3.3333291	1.4166640	4.7123718	3.3352635	1.4166486
10	4.7123725	3.3333293	1.4166642	4.7123853	3.3352836	1.4166647
15	4.7123738	3.3333305	1.4166659	4.7123756	3.3352894	1.4166772
20	4.7123820	3.3333389	1.4166762	4.7123748	3.3353041	1.4166926
25	4.7124300	3.3333887	1.4167367	4.7124173	3.3354278	1.4168363
30	4.7126426	3.3336235	1.4170019	4.7126410	3.3359464	1.4176398
35	4.7128471	3.3338936	1.4173015	4.7128353	3.3364247	1.4186746
40	4.7123918	3.3333925	1.4168063	4.7123946	3.3355951	1.4170695
45	4.7122269	3.3332492	1.4166644	4.7122273	3.3352364	1.4166637
50	4.7121696	3.3332142	1.4166420	4.7121567	3.3351175	1.4165797
55	4.7121709	3.3332178	1.4166412	4.7121400	3.3350847	1.4165527
60	4.7122129	3.3332422	1.4166475	-	-	-
70	4.7123150	3.3332917	1.4166608	-	-	-
80	4.7123729	3.3333156	1.4166672	-	-	-

comparison of the values of the invariants obtained by the present method with those obtained in Ref. [4]. The analytical invariants can be found by using equation(3.3) as

$$I_1 = -3,1415927, \qquad I_2 = 13.3333333, \qquad I_3 = 22.6666667.$$

At $t = 55$, the amplitude of the smaller wave is 0.973752 at the point $x = 52.5$, whereas the amplitude of the larger one is -2.002144 at the point $x = 123.5$. It is found that the absolute difference in amplitudes is 2.62×10^{-2} for the smaller wave and 2.14×10^{-3} for the larger wave. It is observed that the obtained values of the invariants remain almost constant during the computer run. These values are found to be very close with the analytical values. The absolute difference between the values of the invariants obtained by the present method at times $t = 0$ and $t = 80$ are $\Delta I_1 = 3.27 \times 10^{-4}, \Delta I_2 = 5.66 \times 10^{-4}, \Delta I_3 = 2.33 \times 10^{-3}$ whereas they are $\Delta I_1 = 5.1 \times 10^{-2}, \Delta I_2 = 1.38 \times 10^{-1}, \Delta I_3 = 5.59 \times 10^{-1}$ in Ref. [4]. The changes of invariants I_1, I_2, I_3 are less than $9.7 \times 10^{-3}\%$, $3.6 \times 10^{-3}\%$, $9.8 \times 10^{-3}\%$ during the run, respectively. The invariants so

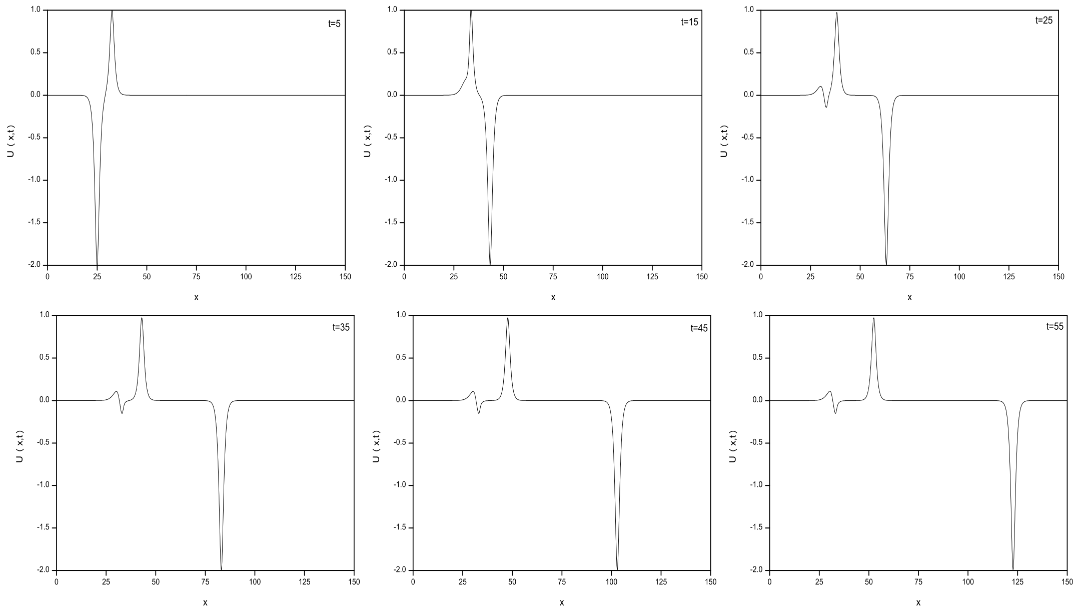

Figure 5: Interaction of two solitary waves with $h = 0.1, \Delta t = 0.025, A_1 = -2, A_2 = 1 (0 \leq x \leq 150)$.

remain constant.

Table 5: Comparison of invariants for the interaction of two solitary waves with results from [4] with $h = 0.1,\quad \Delta t = 0.025$ in the region $0 \leq x \leq 150$.

	Present method			Galerkin method [4]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	-3.1415739	13.3332982	22.6665313	-3.1415915	13.3411364	22.6666177
5	-3.1415356	13.3332986	22.6665162	-3.1373341	13.3297086	22.6211074
10	-3.1294939	13.3157754	22.5691062	-3.1165140	13.2819575	22.3386157
15	-3.1429689	13.3321885	22.6660668	-3.1243642	13.2879992	22.4502917
20	-3.1418073	13.3334504	22.6675388	-3.1190016	13.2781110	22.4081976
25	-3.1416415	13.3335955	22.6678269	-3.1147243	13.2672538	22.3644947
30	-3.1416683	13.3336485	22.6680086	-3.1106562	13.2563740	22.3211768
35	-3.1417128	13.3336930	22.6681812	-3.1065564	13.2454531	22.2776978
40	-3.1417595	13.3337361	22.6683524	-3.1025255	13.2346133	22.2346619
45	-3.1418065	13.3337788	22.6685235	-3.0985577	13.2238575	22.1921206
50	-3.1418535	13.3338213	22.6686945	-3.0945539	13.2130371	22.1493051
55	-3.1419006	13.3338637	22.6688655	-3.0905294	13.2023061	22.1067310

4 Conclusion

In this study, sextic B-spline subdomain procedure for the numerical solution of the MEW equation is presented. According to the two test problems, the comparison calculations with the analytic solution shows that a sextic B-spline subdomain method is capable of solving MEW equation accurately and reliably. The stability analysis of the method is shown to be unconditionally stable. It is also observed that the conservation laws are reasonably well satisfied for the interaction of two solitary waves as well as single solitary wave. The obtained results show that the method can be also used efficiently for solving a large number of physically important non-linear partial differential equations.

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